Short Notes

On equatorial characterization of zonoids and intersection bodies

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Abstract

This study aimed to prove that there is no local equatorial characterization of zonoids in odd dimensions, this gives a negative answer to the conjecture posed by W. Weil in 1977 and shows that the local equatorial characterization of zonoids may be given only in even dimensions. In addition we prove a similar result for intersection bodies, we used the descriptive- deductive method, the study found that there is no local characterization of these bodies.

Keywords: *Equatorial characterization, zonoids, intersection bodies*

المستخلص

هدفت هذه الدراسة إلى إثبات إنه ليس هناك معيارللتساوى الموضعى للزونويد فى الأبعاد الفردية، هذا يعطى إجابة سلبية للتخمين الذى طرحه و.وييل فى ١٩٧٧ و يبين إن المعيار للتساوى الموضعى للزونويد يمكن أن يعطى فقط فى حالة الأبعاد الزوجية. بالإضافة إلى إننا أثبتنا نتيجة مماثلة للأجسام المتقاطعة، إستخدمنا المنهج الوصفى– الإستنباطى، توصلت الدراسة إلى إنه ليس هناك معيار موضعى لهذه الإجسام.

كلمات مفتاحية: معيار تساوى ، زونويد ، أجسام متقاطعة

Introduction

A zonoid in $\mathbb{R}^n$ is an origin symmetric convex body that can be approximated (in the Hausdorff metric) by finite Minkowski sums of line segments. It turns out that zonoids appear in many different contexts in convex geometry, physics, optimal control theory, and functional analysis (see [1], [3], [4]). One of the equivalent definitions of zonoids, useful in convex geometry, leads to a notion of a projection body. An origin symmetric convex body L in $\mathbb{R}^n$ is called a projection body is there exists another origin symmetric convex body K such that the support function of L in every direction is equal to the volume of the hyperplane projection of K orthogonal to this direction: for every

$$\lambda \in T^{n-1}, h_L(\lambda) = \mathrm{Vol}_{n-1}(K|\lambda^{\perp}), \lambda^{\perp} = \{y \in \mathbb{R}^n : \lambda . y = 0\}.$$

The support function $h_L(\lambda) = \max_{x \in L} \lambda . x$ is equal to the dual norm $\|\lambda\|_{L^*}$ where L^* stands for the polar body of L. From the above definition and Cauchy formula (see [15]), we immediately derive the following analytic definition, which will be useful for us in this study: An origin symmetric convex body $L \subset \mathbb{R}^n$ is a zonoid if and only if

$$h_L(\lambda) = Cos\mu(\lambda) := \int_{T^{n-1}} |\lambda . \theta| d\mu(\theta)$$

with some even positive measure μ on T^{n-1}. Finally, a functional analytic definition shows that an origin symmetric convex body $L \subset \mathbb{R}^n$ is a zonoid if and only if it is a polar body to the unit ball of a subspace of L_1.

It is well known that every origin symmetric convex body in $\mathbb{R}^n$ is projection body, but this is no longer true in $\mathbb{R}^n$ for $n \geq 3$ (see [26],

[15]). It is an interesting question how to determine if a given convex body is a zonoid or not. It is very reasonable to assume that one can provide a strictly local characterization of zonoids. This question was posed repeatedly (see [26] , for the history of problem), however [28] showed that a local characterization of zonoids does not exist. In particular, he showed that there exists an origin symmetric convex C^{∞} body $K \subset \mathbb{R}^n, n \geq 3$, that is not a zonoid but has the following property: for every $u \in T^{n-1}$ there exists a zonoid Z_u centered at the origin and a neighborhood $U_u \subset T^{n-1}$ of u such that the boundaries of K and Z_u coincide at all points where the exterior unit normal vectors belong to U_u. Thus, no characterization of zonoids that involves only arbitrarily small neighborhoods of boundary points is possible.(see [28]) ,Weil proposed the following conjecture about local equatorial characterization of zonoids. Let $L \subset \mathbb{R}^n$ be an origin symmetric convex body and assume that for any equator $\varphi \subset T^{n-1}$, there exists a zonoid Z_{φ} and a neighborhood E_{φ} of φ such that the boundaries of L and Z_{φ} coincide at all points where the exterior unit vector belongs to

E_φ ; then L is a zonoid. Affirmative answers for even dimensions were given independently by [23], [13] but the question was left open in odd dimensions. That was a consequence of the fact that the inversion formulas for the cosine transform are not local in odd dimensions (see [22]).

In this study we show that the answer to the conjecture in odd dimensions is negative. We prove that in both cases (for odd and even dimensions) the answer can be obtained as a consequence of the characterization of zonoids in terms of sections of the polar body, given in [18]. In even dimensions the answer follows directly from the geometric inversion formula for Cosine transform [18]. The odd dimensional case,on the other hand, requires much more tricky and detailed analysis of the behavior of the inverse Cosine transform.

Our main tool is the Fourier analytic inversion formula from [8] or [15]. It allows to obtain the results for zonoids together with the results about the intersection bodies. The notion of an intersection body of star body was introduced by [21]. K is called the intersection body of L if the radius of K in every direction is equal to the $(n-1)$–dimensional volume of the central hyperplane section of L perpendicular to this direction: $\forall\lambda \in T^{n-1}$

$$\rho_k(\lambda) = \mathrm{Vol}_{n-1}(L \cap \lambda^{\perp}), \lambda^{\perp},$$

where $\rho_k(\lambda) = \max\{a: a\lambda \in K\}$ is the radial function of body K. Passing to polar coordinates in $\lambda^{\perp}$, we derive the following analytic definition of an intersection body of star body: K is called the intersection body of L if

$$\rho_k(\lambda) = \frac{1}{n-1}\psi\rho_L^{n-1}(\lambda) := \frac{1}{n-1}\int_{T^{n-1}\cap\lambda^{\perp}} \rho_L^{n-1}(\theta)d\theta.$$

Here ψ stands for the spherical Radon transform (see [22]).

A more general class of intersection bodies was defined by ([5] and [29] as the closure of intersection bodies of star bodies in the radial metric

$$d(K, L) = \sup_{\lambda\in T^{n-1}}|\rho_k(\lambda) - \rho_L(\lambda)|.$$

In this study we will consider only C^∞ smooth intersection bodies: a body K is an intersection body if there exists an even nonnegative function f on T^{n-1}, such that the radial function of K is a spherical Radon transform ψf of f. Since we can always define

$L{:}\,\rho_L^{n-1}(\theta) = (n-1)f(\theta),$

we will not distinguish between intersection bodies of star bodies and intersection bodies. We prove that the local equatorial characterization of intersection bodies is not possible in odd dimensions. Namely, we show that one can construct an origin symmetric convex body $L \subset \mathbb{R}^n, n \geq 5$ is odd, such that for any equator

$\varphi \subset T^{n-1}$, there exists an intersection body I_φ and a neighborhood E_φ of φ such that the boundaries of L and I_φ coincide at all points of E_φ (i.e, $\rho_L(\lambda) = \rho_{I_\varphi}(\lambda)$ for all $\lambda \in E_\varphi$); but nevertheless, L is not an intersection body. On the other hand, we show that the local equatorial characterization of intersection bodies is possible in even dimensions.

We also extend the result of [28] to the class of intersection bodies by proving that there is no local characterization of those bodies in odd and even dimensions.We prove that there exists an origin symmetric convex C^∞ body $K \subset \mathbb{R}^n, n \geq 5$, that is not an intersection body, but has the following property : for each $u \in T^{n-1}$ there exists an intersection body I_u centered at the origin and a neighborhood $U_u \subset T^{n-1}$ of u such that the boundaries of K and I_u coincide on U_u. In odd dimensions this is a consequence of the lack of a local equatorial characterization of intersection bodies mentioned above but we give an independent proof that does not distinguish between even and odd dimensions.

Our proofs for zonoids and intersection bodies are very similar, they are based on almost identical Fourier analytic inversion formulas for the Cosine and Radon transforms. This is one more indication of the remarkable duality between sections and projections (see[19]) .

1. Auxiliary results

Our main tool is the Fourier transform of distributions (see [9],[10] and [15] for exact definitions and properties) and the connections between the Cosine and the spherical Radon transforms and the Fourier transform.

We start with the connection of the spherical Radon transform and the Fourier transform. A Koldobsky (see for example Lemma 3.7 in [15]) proved that

$$\psi g(\lambda) = \frac{1}{\pi}\hat{g}(\lambda), \qquad \forall \lambda \in T^{n-1}, \qquad (1)$$

provided that g is an even homogeneous function of degree $-n+1$ on $\frac{\mathbb{R}^n}{\{0\}}.n>1$, satisfying $g|_{T^{n-1}} \in L_1(T^{n-1})$.

An immediate consequene of this formula is the following Fourier analytic a characterization of intersection bodies (see [15] Theorem 4.1): An origin symmetric star body K is an intersection body if and only if ρ_K, extended to $\mathbb{R}^n$ as a homogeneous function of degree -1, represents a positive definite distribution on $\mathbb{R}^n$. When K is infinitely smooth, this is equivalent to $\widehat{\rho_K} \geq 0$.

A very similar connection of the Cosine transform and the Fourier transform was established in ([18] see also [15]):

$$Cosg(\lambda) = \frac{2}{\pi}\hat{g}(\lambda), \qquad \forall \lambda \in T^{n-1}, \qquad (2)$$

provided that g is an even homogeneous function of degree $-n-1$

on $\frac{\mathbb{R}^n}{\{0\}}.n>1$, satisfying $g|_{T^{n-1}} \in L_1(T^{n-1})$.

As above, one can obtain a very similar Fourier analytic a characterization of zonoids (see [15] Theorem 8.6): An origin symmetric star body K is a zonoid if and only if h_K, extended to $\mathbb{R}^n$ as homogeneous function of degree 1, represents a negative definite distribution on $\mathbb{R}^n$. When K is infinitely smooth, this is equivalent to $\widehat{h_K} \leq 0$.

Our next tool is a formula connecting the Fourier transform of powers of the radial function with the derivatives of the parallel section function. Let D be an infinitely smooth origin symmetric star body in $\mathbb{R}^n, \lambda \in T^{n-1}$, and let

$\lambda^{\perp} = \{x \in \mathbb{R}^n : x.\lambda = 0\}$. We denote by

$$A_{D,\lambda}(s) = \mathrm{Vol}_{n-1}(D \cap \{\lambda^{\perp} + s\lambda\}), s \in \mathbb{R},$$

The parallel section function of D in the direction of λ. The following formula was proved in [8] see [15]: for any $\lambda \in T^{n-1}$ and $k \in \mathbb{N}, k \neq n-1$,

$$\widehat{\rho_D^{n-k-1}}(\lambda) = (-1)^{\frac{k+1}{2}}\pi(n-k-1)A_{D,\lambda}^{k}(0), \qquad (3)$$

when k is even, and

$$\widehat{\rho_D^{n-k-1}}(\lambda) = (-1)^{\frac{k+1}{2}}2(n-k-1)k!\int_0^{\infty}\frac{A_{D,\lambda}(z) - A_{D,\lambda}(0) - \cdots - A_{D,\lambda}^{(k-1)}(0)\frac{z^{k+1}}{(k-1)!}}{z^{k+1}}dz, \qquad (4)$$

when k is odd.

As a consequence of equations (1), (3) and (4) with $k = n-2$, we obtain the

Fourier analytic a characterization of intersection bodies (see [15]).

Let L be an origin symmetric star body in $\mathbb{R}^n$ such that ρ_L is infinitely differentiable on $\lambda \in T^{n-1}$. The body L is an intersection body if and only if $\forall \lambda \in T^{n-1}$,

$$(-1)^{\frac{n-2}{2}} A_{L,\lambda}^{(n-2)}(0) \geq 0, \tag{5}$$

when n is even, and

$$(-1)^{\frac{n-1}{2}} \int_0^\infty \frac{A_{L,\lambda}(z) - A_{L,\lambda}(0) - \cdots - A_{L,\lambda}^{(n-3)}(0)\frac{z^{n-3}}{(n-3)!}}{z^{n-1}} dz \geq 0, \tag{6}$$

when n is odd.

Similarly, using the duality relation $h_D = \rho_{D^*}^{-1}$ and equations (2), (3) and (4) with $k = n$, one can obtain the following characterization of of zonoids (see [18] or [15]). Let L be an origin symmetric convex body in $\mathbb{R}^n$ such that h_L is infinitely differentiable on T^{n-1}. The body L is a zonoid (projection body) if and only if $\forall \lambda \in T^{n-1}$,

$$(-1)^{\frac{n}{2}} A_{L^*,\lambda}^{(n)}(0) \geq 0, \tag{7}$$

when n is even, and

$$(-1)^{\frac{n+1}{2}} \int_0^\infty \frac{A_{L^*,\lambda}(z) - A_{L^*,\lambda}(0) - \cdots - A_{L^*,\lambda}^{(n-1)}(0)\frac{z^{n-1}}{(n-1)!}}{z^{n+1}} dz \geq 0, \tag{8}$$

when n is odd.

2. There is no local equatorial characterization of intersection bodies in odd dimensions.

To construct counterexample, it is natural to use (6). This formula shows that one has to use the information about the section function $A_{L,\lambda}(z)$ of the body along the whole range of z. For $0 < \epsilon < 1$ and $\lambda \in T^{n-1}$, we denote by $U_\epsilon(\lambda)$ the union of caps centered at λ and $-\lambda$:

$$U_\epsilon(\lambda) \coloneqq \left\{\theta \in T^{n-1} \colon |\theta.\lambda| \geq \sqrt{1-\epsilon^2}\right\}.$$

We denote by $E_\epsilon(\lambda)$, $0 < \epsilon < 1$, the neighborhood of the equator $T^{n-1} \cap \lambda^\perp$:

$$E_\epsilon(\lambda) \coloneqq \{\theta \in T^{n-1} \colon |\theta.\lambda| < \epsilon\}.$$

The following result is crucial for the construction of the counterexample. Its proof is based on the fact that the inversion formula (6) is not local.

Lemma 3.1. Let $n \geq 3$ be odd. Then there exists an $\epsilon > 0$ and an absolute constant $c > 0$ such that for any $x, \lambda \in T^{n-1}$, there exists an even function $f_{x,\lambda}$ satisfying $f_{x,\lambda} = 0$ on $E_\epsilon(x)$, and

$$\psi^{-1} f_{x,\lambda} > c \text{ on } U_\epsilon(\lambda).$$

Proof. First, we fix $x, \lambda \in T^{n-1}$ and find $\epsilon = \epsilon(x, \lambda)$ and $c = c(x, \lambda)$ satisfying the requirement of the lemma. Then we use the compactness argument to produce absolute ϵ and c.

For fixed $x, \lambda \in T^{n-1}$ and some small $\epsilon > 0$ we take two auxiliary infinitely smooth symmetric star bodies M, Q, such that $\rho_M = \rho_Q$ on the closure of $E_\epsilon(\lambda) \cup E_\epsilon(x)$, and $\rho_M > \rho_Q$ otherwise. We put $f_{x,\lambda} = (-1)^{\frac{n-1}{2}}(\rho_M - \rho_Q)$. Then $f_{x,\lambda} = 0$ on $E_\epsilon(x)$, and $\rho_M = \rho_Q$ on $E_\epsilon(\lambda)$ implies

$$A^k_{M,\lambda}(0) = A^k_{Q,\lambda}(0), k = 0,1, \dots, n-3.$$

Thus (1) and (4) with $k = n-2$ imply

$$\psi^{-1} f_{x,\lambda}(\lambda) = (-1)^{\frac{n-1}{2}}\left(\psi^{-1}\rho_M(\lambda) - \psi^{-1}\rho_Q(\lambda)\right) = (-1)^{n-1}(2\pi)^{n-1}(n-2)! \int_0^\infty \frac{A_{M,\lambda}(z) - A_{Q,\lambda}(0)}{z^{n-1}} dz > 0,$$

since $Q \subseteq M$. We proved that for fixed $x, \lambda \in T^{n-1}$ there exists $\epsilon' = \epsilon'(x,\lambda) > 0$ and $c' = c'(x,\lambda) > 0$ such that there exists an even function $f_{x,\lambda}$ satisfying $f_{x,\lambda} = 0$ on $E_\epsilon(x)$, and $\psi^{-1} f_{x,\lambda}(\lambda) \geq c'$. The function $\psi^{-1} f_{x,\lambda}$ is continuous on T^{n-1} since M, Q are infinitely smooth (see [15] Lemma 2.4). Hence,

$\psi^{-1} f_{x,\lambda} \geq c > 0$ on $U_{\epsilon''}(\lambda)$, for some $\epsilon'' > 0$ and $c = c(x, \lambda)$.

Put $\tilde{\epsilon} = \tilde{\epsilon}(x,\lambda) = \min(\epsilon', \epsilon'')$. We prove that for any x and λ, there is

$\tilde{\epsilon} = \tilde{\epsilon}(x,\lambda) > 0$ and a function $f_{x,\lambda}$ such that $f_{x,\lambda} = 0$ on $E_{\tilde{\epsilon}}(x)$, but $\psi^{-1} f_{x,\lambda} \geq c$ on $U_{\tilde{\epsilon}}(\lambda), c = c(x,\lambda)$.

Now we use the compactness argument to show that we can choose $\tilde{\epsilon}$ and c independent of x and λ. We choose a finite set of pairs $\{x_i, \lambda_i\}_{i=1}^m$ such that $\left\{U_{\frac{\tilde{\epsilon}}{2}}(x_i) \times U_{\frac{\tilde{\epsilon}}{2}}(\lambda_i)\right\}_{i=1}^m$ cover $T^{n-1} \times T^{n-1}$. We take

$$\epsilon = \frac{1}{2} \min_{1 \leq i \leq m} \tilde{\epsilon}_i \text{ and } c = \min_{1 \leq i \leq m} c(x_i, \lambda_i).$$

Then, for any (x, λ), there is a pair (x_i, λ_i) such that

$(x,\lambda) \in U_{\frac{\tilde{\epsilon}}{2}}(x_i) \times U_{\frac{\tilde{\epsilon}}{2}}(\lambda_i)$ and thereby

$$E_\epsilon(x) \times U_\epsilon(\lambda) \subset E_{\tilde{\epsilon}_i}(x_i) \times U_{\tilde{\epsilon}_i}(\lambda_i).$$

Finally, we may define $f_{x,\lambda} = f_{x_i,\lambda_i}$ (see [22]).

Remark 3.2. Note that, dilating M and Q (and thus functions $f_{x,\lambda}$), we may assume that c is as large as we want. By the technical resons that will become clear later, we take $c = 2\psi^{-1}1$. Moreover, we can assume that the set of functions $\{f_{x,\lambda}\}_{x,\lambda \in T^{n-1}}$ in the lemma is finite.

Let C_+^∞ be the class of origin symmetric convex bodies with C^∞ boundary and everywhere positive Gaussian curvature (see [6]). The following auxiliary result seems to be well-known. It is interesting to note that it is not true without the C_+^∞ assumption though (see [26] [14], [20], [2]).

Lemma 3.3. Let $M \in C_+^\infty$ and let

$K(s) = sB_2^n + (1-s)M$ be Minkowski sum of sB_2^n and $(1-s)M, s \in [0,1]$. Then the map

$s \to \psi^{-1}\rho_{k(s)}(\lambda), \lambda \in T^{n-1}$, is continuous.

Proof. We note first that for any fixed $s \in [0,1]$, the boundary $\partial K(s)$ of $K(s)$ is C^∞. Indeed, $\partial K(s)$ can be parameterized as

$$u \in T^{n-1} \to \nabla h_{(1-s)M}(u) + su$$
$$= (1-s)\nabla h_M(u) + su,$$

where $u \in T^{n-1} \to (1-s)\nabla h_M(u)$

is a parameterization of $(1-s)\partial M$. Here

$$\nabla h_{(1-s)M}(u) = v^{-1}(u),$$

and $v: (1-s)\partial M \to T^{n-1}$

is the spherical image map (see [6] ,[26]), to show that the map $u \in T^{n-1} \to \nabla h_M(u)$ is a C^∞ diffeomorphism. Hence, the map

$$u \in T^{n-1} \to g_s(u) := (1-s)\nabla h_M(u) + su$$

is also a C^∞ diffeomorphism. To show that $s \to \psi^{-1}\rho_{k(s)}(\lambda)$ is continuous, we pick any $s \in [0,1]$ and take any sequence $\{s_m\}_{m=1}^\infty$ of points from $[0,1]$ converging to s. The map

$$u \in T^{n-1} \to f_s(u) := \frac{g_s(u)}{|g_s(u)|}$$

is a C^∞ diffeomorphism for any $s \in [0,1]$, and $f_{s_m} \to f_s$ in $C^\infty(T^{n-1})$. Hence, $f_{s_m}^{-1} \to f_s^{-1}$ in $C^\infty(T^{n-1})$.

Now, $g_s\left(f_s^{-1}(\lambda)\right) \in \partial K(s)$ implies $\rho_{k(s)}(\lambda) = \left|g_s\left(f_s^{-1}(\lambda)\right)\right|$, and $\rho_{k(s_m)}$ converges to $\rho_{k(s)}$ in $C^\infty(T^{n-1})$. Since ψ is a continuous bijection of $C^\infty(T^{n-1})$ to itself, (see [6]), the lemma is proved.

Lemma 3.4. Let $n \geq 5$. For any point $\lambda_0 \in T^{n-1}$ there exists $\widetilde{K} \in C_+^\infty$ such that $\psi^{-1}\rho_{\widetilde{K}}(\lambda)$ is strictly positive for all

$\lambda \neq \pm\lambda_0$ and $\psi^{-1}\rho_{\widetilde{K}}(\pm\lambda_0) = 0$.

Proof. Fix $n \geq 5$. Then there exists $M \in C_+^\infty$ such that $\psi^{-1}\rho_M(\lambda)$ is sign-changing

(see [15] Lemma 4.10 where an example of such body is constructed). For $s \in [0,1]$, consider the Minkowski sum $K(s) = sB_2^n + (1-s)M$. Then $\psi^{-1}\rho_{K(0)}(\lambda)$ is sign-changing and there exists $\Lambda' \subset T^{n-1}$ such that

$\psi^{-1}\rho_{K(0)}(\lambda) < 0, \forall\lambda \in \Lambda'$. On the other hand , $\psi^{-1}\rho_{K(1)}(\lambda) > 0, \forall\lambda \in T^{n-1}$. By the previous lemma the map $s \to \psi^{-1}\rho_{K(s)}(\lambda)$ is continuous, and there is

$s_0 \in [0,1]$ such that

$\psi^{-1}\rho_{K(s_0)}(\lambda) \geq 0, \forall\lambda \in T^{n-1}$ and

$\psi^{-1}\rho_{K(s_0)}(\lambda) = 0, \forall\lambda \in \Lambda \subset T^{n-1}$,

for some $\Lambda \neq \emptyset$. Fix any $\lambda_0 \in \Lambda$. Consider an even C^∞ smooth function g on T^{n-1} such that $g(x) > 0, \forall x \neq \pm\lambda_0$ and $g(\pm\lambda_0) = 0$. For $\epsilon > 0$ define a body $\widetilde{K}$ (depending on λ_0): $\psi^{-1}\rho_{\widetilde{K}}(\lambda) = \psi^{-1}\rho_{K(s_0)}(\lambda) + \epsilon g(\lambda)$.

Note that $\psi^{-1}\rho_{\widetilde{K}}(\lambda)$ is strictly positive for all

$\lambda \neq \pm\lambda_0$, and $\psi^{-1}\rho_{\widetilde{K}}(\pm\lambda_0) = 0$. We get

$\rho_{\widetilde{K}}(x) = \rho_{K(s_0)}(x) + \epsilon\psi g(x)$.

Since ψg is a C^∞ function, and $K(s_0) \in C_+^\infty$, we may choose ϵ small enough so that $\widetilde{K} \in C_+^\infty$. Using the rotation argument, we can take λ_0 to be arbitrary.

Theorem 3.5. Let $n \geq 5$ be odd. There exists $\epsilon > 0$ and a conver symmetric body K that is not an intersection body, but nevertheless $\forall x \in T^{n-1}$ there exists an intersection body L_x such that $\rho_K = \rho_{L_x}$ on $E_\epsilon(x)$.

Proof. We define a convex body K and a family of convex bodies $\{L_x\}_{x\in T^{n-1}}$ using $\widetilde{K}$ and functions f_{x,λ_0} from Lemma 3.1. We fix some small ϵ satisfying the requirements of Lemma 3.1 and we may assume $c = 2\psi^{-1}1$ (see Remark 3.2). Then, define $K = K_{\delta,\lambda_0}$ via $\rho_K = \rho_{\widetilde{K}} - \delta$, where for the moment $\delta > 0$ is assumed to be so small that $K \in C_+^\infty$ and $\psi^{-1}\rho_K$ is strictly positive outside $U_\epsilon(\lambda_0)$. Note that $\psi^{-1}\rho_K(\lambda_0) < 0$ and thus K is not an intersection body.

Now we define a family of convex bodies $\{L_x\}_{x\in T^{n-1}}$. Since $\widetilde{K} \in C_+^\infty$, we take δ so small that

$\rho_{L_x} := \rho_{\widetilde{K}} - \delta + \delta f_{x,\lambda_0} > 0$ on T^{n-1}

and L_x is convex. Observe that

$\rho_{L_x} := \rho_K$ on $E_\epsilon(x)$ for any $x \in T^{n-1}$. We can assume that δ is so small that

$\psi^{-1}\rho_{L_x} = \psi^{-1}\rho_{\widetilde{K}} - \delta\psi^{-1}1 + \delta\psi^{-1}f_{x,\lambda_0} > 0$

on $\frac{T^{n-1}}{U_\epsilon(\lambda_0)}$, since $\psi^{-1}\rho_{\widetilde{K}} > 0$ on $\frac{T^{n-1}}{U_\epsilon(\lambda_0)}$.

To show that bodies L_x are intersection bodies $\forall x \in T^{n-1}$, it is enough to prove that

$\psi^{-1}\rho_{L_x} > 0$ on $U_\epsilon(\lambda_0)$. By Remark 3.2,

$$\min_{x\in T^{n-1}} \psi^{-1}f_{x,\lambda_0} \geq 2\psi^{-1}1 \quad \text{on} \quad U_\epsilon(\lambda_0),$$

hence

$$\psi^{-1}\rho_{L_x} = \psi^{-1}\rho_{\widetilde{K}} - \delta\psi^{-1}1 + \delta\psi^{-1}f_{x,\lambda_0} \geq \delta\psi^{-1}1 > 0$$

on $U_\epsilon(\lambda_0)$. Moreover, $\delta > 0$ can be chosen independently of x since the set of functions $\{f_{x,\lambda}\}_{x,\lambda\in T^{n-1}}$ in Lemma 3.1 is finite.

3. There is no local equatorial characterization of of zonoids in odd dimensions.

The proofs in this section are very similar (in fact, almost identical) to the ones in the previous section.

Lemma 4.1. Let $n \geq 3$ be odd. Then there exists an $\epsilon > 0$ and an absolute constant $c > 0$ such that for any $x, \lambda \in T^{n-1}$, there exists an even function $f_{x,\lambda}$ satisfying $f_{x,\lambda} = 0$ on $E_\epsilon(x)$, and $Cos^{-1}f_{x,\lambda} \geq c$ on $U_\epsilon(\lambda)$.

Proof. The proof follows the same lines as that of Lemma 3.1. One has to change the Spherical Radon transform to the Cosine transform, put support functions instead of radial functions and thus, use section functions of polar bodies together with (2), (4) and (8).

Remark 4.2. Note that dilating M and Q (and thus functions $f_{x,\lambda}$) we may assume that c is as large as we want. For technical reasons, we take $c = 2Cos^{-1}1$. Moreover, we can assume that the set of functions $\{f_{x,\lambda}\}_{x,\lambda\in T^{n-1}}$ in the lemma is finite.

Lemma 4.3. Let $n \geq 3$. For any point $\lambda_0 \in T^{n-1}$ there exists a zonoid $\widetilde{K} \in C_+^\infty$ such that $Cos^{-1}h_{\widetilde{K}}(\lambda)$ is strictly positive for all

$\lambda \neq \pm\lambda_0$ and $Cos^{-1}h_{\widetilde{K}}(\pm\lambda_0) = 0$.

Proof. Fix $n \geq 3$. Then there exists $M \in C_+^\infty$ such that $Cos^{-1}h_M$ is sign-changing

(see [15], the Fourier Analytic solution of Shephard problem for a construction of a C_+^∞ non-zonoid body).

For $s \in [0,1]$ consider the Minkowski sum $K(s) = sB_2^n + (1-s)M$. Then $h_{K(s)} =$ th $B_2^n + (1-s)h_M$ is a C^∞ -function, $Cos^{-1}h_{K(0)}(\lambda)$ is sign-changing and there exists $\Lambda' \subset T^{n-1}$ such that $Cos^{-1}h_{K(0)}(\lambda) < 0, \forall\lambda \in \Lambda'$. On the other hand, $Cos^{-1}h_{K(1)}(\lambda) > 0, \forall\lambda \in T^{n-1}$. The map $s \to Cos^{-1}h_{K(s)}$ is continuous, since Cos is a continuous bijection of $C^\infty(T^{n-1})$ into itself. (see [6]). Hence, there is $s_0 \in [0,1]$ such that

$Cos^{-1}h_{K(s_0)} \geq 0$, and $Cos^{-1}h_{K(s_0)}(\lambda) = 0$, $\forall\lambda \in T^{n-1}$ and some $\Lambda \neq \emptyset$. Fix any $\lambda_0 \in \Lambda$. Consider an even C^∞ smooth function g on T^{n-1} such that

$g(x) > 0, \forall x \neq \pm\lambda_0$ and $g(\pm\lambda_0) = 0$. For $\epsilon > 0$ define a body $\widetilde{K}$:

$$Cos^{-1}h_{\widetilde{K}}(\lambda) = Cos^{-1}h_{K(s_0)}(\lambda) + \epsilon g(\lambda).$$

Note that $Cos^{-1}h_{\widetilde{K}}(\lambda)$ is strictly positive for all $\lambda \neq \pm\lambda_0$, and

$Cos^{-1}h_{\widetilde{K}}(\pm\lambda_0) = 0$. Moreover, $h_{\widetilde{K}} = h_{K(s_0)} + \epsilon Cosg$.

Since $Cosg$ is a continuous function and $K(s_0) \in C_+^\infty$, we may choose ϵ small enough so that $\widetilde{K} \in C_+^\infty$. Using the rotation argument, we can take λ_0 to be arbitrary.

Theorem 4.4. Let $n \geq 3$ be odd.There exists $\epsilon > 0$ and a convex body K that is not a zonoid, but nevertheless $\forall x \in T^{n-1}$ there exists a zonoid L_x such

$$h_K = h_{L_x} \text{ on } E_\epsilon(x).$$

Proof. We define a convex body K and a family of convex bodies $\{L_x\}_{x\in T^{n-1}}$ using the zonoid $\widetilde{K}$ and functions f_{x,λ_0} from Lemma 4.1. We fix some small ϵ satisfying the requirements of Lemma 4.1 with $c = 2Cos^{-1}1$ (see Remark 4.2). Then, define $K = K_{\delta,\lambda_0}$ via $h_K = h_{\widetilde{K}} - \delta$, where for the moment $\delta > 0$ is assumed to be so small that $K \in C_+^\infty$ and $Cos^{-1}h_K$ is strictly positive outside $U_\epsilon(\lambda_0\,) < 0$ and thus K is not a zonoid. Now we define a family of convex bodies $\{L_x\}_{x\in T^{n-1}}$. Since $\widetilde{K} \in C_+^\infty$, we take δ so small that $h_{L_x} \coloneqq h_{\widetilde{K}} - \delta + \delta f_{x,\lambda_0} > 0$ on T^{n-1} and L_x is convex. Observe that $h_{L_x} = h_K$on $E_\epsilon(x)$ for any $x \in T^{n-1}$. We can assume that δ is so small that

$$Cos^{-1}h_{L_x} = Cos^{-1}h_{\widetilde{K}} - \delta Cos^{-1}1 + \delta Cos^{-1}f_{x,\lambda_0} > 0 \text{ on } \frac{T^{n-1}}{U_\epsilon(\lambda_0\,)}, \text{ since } Cos^{-1}h_{\widetilde{K}} > 0 \text{ on } \frac{T^{n-1}}{U_\epsilon(\lambda_0\,)}.$$

To show that bodies L_x are zonoids $\forall x \in T^{n-1}$, it is enough to prove that $Cos^{-1}h_{L_x} > 0$ on $U_\epsilon(\lambda_0\,)$. By Remark 4.2, $\min_{x\in T^{n-1}} Cos^{-1}f_{x,\lambda_0} > 2Cos^{-1}1$ on $U_\epsilon(\lambda_0\,)$, hence $Cos^{-1}h_{L_x} = Cos^{-1}h_{\widetilde{K}} - \delta Cos^{-1}1 + \delta Cos^{-1}f_{x,\lambda_0} \geq \delta Cos^{-1}1 > 0$

on $U_\epsilon(\lambda_0\,)$, and the result follows.

4. There is a local equatorial characterization of intersection bodies and zonoids in even dimensions.

We consider at first intersection bodies. The proof of the following lemma is obtained by a straightforward repetition of the argument from (see [15]), and we omit the details.

Lemma 5.1. Let $g(x)$ be an even homogeneous function of degree-1 such that $g(x)$ is nonnegative and infinitely smooth on T^{n-1}. Then

$$\tilde{g}(\lambda) = (-1)^{\frac{n-2}{2}} \pi A_{g,\lambda}^{n-2}(0),$$

where

$$A_{g,\lambda}(z) = \int_{\{y\in\mathbb{R}^n : y.\lambda = z\}} \chi_{[0,1]}\left(\frac{1}{g(y)}\right) dy, \lambda \in T^{n-1}.$$

Theorem 5.2. Let n be even and let $K \subset \mathbb{R}^n$ be an origin symmetric convex body. Assume that for any great sphere $\lambda^{\perp} \cap T^{n-1}$, there exists an intersection body L_λ and a neighborhood $E_{\epsilon(\lambda)}(\lambda)$ of $\lambda^{\perp} \cap T^{n-1}$ such that the radial functions of K and L_λ coincide at all points of $E_{\epsilon(\lambda)}(\lambda)$; then K is an intersection body

Proof. If K and L_λ are infinitely smooth, then it is enough to observe that

$\rho_K(u) = \rho_{L_\lambda}(u)\ \forall u \in E_{\epsilon(\lambda)}(\lambda)$

implies $A_{K,\lambda}(s) = A_{L_\lambda,\lambda}(s)$ for sufficiently small s and apply (5).

Consider the general case case. It was proved by A. Koldobsky that an original-symmetric convex body K is an intersection body if and only if ρ_K represents a positive definite distribution (see for example, Theorem 4.1 in [15]). Thus, it is enough to show that $\langle \widehat{\rho_K}, \eta \rangle \geq 0$, for all nonnegative test functions η on $\mathbb{R}^n$.

Using the definition of the Fourier Transform of distributions (see section 2.5, in [15]) and passing to the polar coordinates, we get

$$\langle \widehat{\rho_K}, \eta \rangle = \langle \rho_K, \hat{\eta} \rangle = \int_{\mathbb{R}^n} \rho_K(x)\, \hat{\eta}(x) dx$$

$$= \int_{T^{n-1}} \rho_K(\theta) \int_0^\infty r^{n-2} \hat{\eta}(r\theta) dr d\theta.$$

Observe that the function

$$\alpha(x) \coloneqq \int_0^\infty r^{n-2} \hat{\eta}(rx) dr, x \in \frac{R^{n-1}}{\{0\}}$$

is homogeneous of degree-$n+1$ and infinitely smooth. Hence, we may apply equality 4.3, page 72 together with Lemma 3.7, page 53 from ([15]) to claim that there exists an infinitely smooth non-negative homogeneous of degree -1 function

$g(x) = \frac{1}{2}\int_{\mathbb{R}} \eta(sx)\, ds$, such that $\hat{g}(\theta) = \alpha(\theta)\ \forall \theta \in T^{n-1}$.

Thus,

$\int_{T^{n-1}} \rho_K(\theta) \int_0^\infty r^{n-2} \hat{\eta}(r\theta) dr d\theta = \int_{T^{n-1}} \rho_K(\theta)\, \hat{\eta}(\theta) d\theta.$

Using a partition of unity on T^{n-1} together with homogeneity of g, we can write

$$g(\theta) = \sum_{j=1}^{m} g_j(\theta) = \sum_{j=1}^{m} \frac{1}{2} \int_{\mathbb{R}} \eta_j\,(s\theta)\, ds, \theta \in T^{n-1},$$

where $\operatorname{supp} g_j|_{T^{n-1}} \subset U_{\epsilon_j}(\lambda_j)$ are small enough.

By the previous lemma, $\operatorname{supp} g_j|_{T^{n-1}} \subset U_{\epsilon_j}(\lambda_j)$ implies $\operatorname{supp} \hat{g}_j|_{T^{n-1}} \subset U_{\epsilon_j}(\lambda_j)$.

Hence,

$$\langle \widehat{\rho_K}, \eta \rangle = \sum_{j=1}^{m} \int_{T^{n-1}} \rho_K(\theta)\, \hat{g}_j(\theta) d\theta$$

$$= \sum_{j=1}^{m} \int_{E_{\epsilon_j}(\lambda_j)} \rho_K(\theta)\, \hat{g}_j(\theta) d\theta$$

$$= \sum_{j=1}^{m} \int_{E_{\epsilon_j}(\lambda_j)} \rho_{L_{\epsilon_j}}(\theta)\, \hat{g}_j(\theta) d\theta$$

$$= \sum_{j=1}^{m} \int_{T^{n-1}} \rho_{L_{\epsilon_j}}(\theta)\, \hat{g}_j(\theta) d\theta$$

$$= \sum_{j=1}^{m} \langle \widehat{\rho_{L_{\epsilon_j}}}, \eta_j \rangle \geq 0.$$

The following result was obtained independently by [23] and[11]. Its proof could be also obtained by the arguments similar to those in the previous proof, and we omit it.

Theorem 5.3. Let n be even and let $K \subset \mathbb{R}^n$ be an origin symmetric convex body. Assume that for any great sphere $\lambda^{\perp} \cap T^{n-1}$, there exists a zonoid Z_λ and a neighborhood $E_{\epsilon(\lambda)}(\lambda)$ of $\lambda^{\perp} \cap T^{n-1}$ such that the boundaries of K and Z_λ coincide at all points where the exterior unit vector belong to $E_{\epsilon(\lambda)}(\lambda)$; then K is a zonoid.

5. There is no local equatorial characterization of intersection bodies .

In this section we prove the analog of the result of [28] for zonoids. Our proof is different from the one of W. Weil. We show that, given $x, \lambda \in T^{n-1}$, one can construct a function f which is zero around x, but such that the inverse spherical Radon transform of f is positive around λ. For convenience of the reader we split the proof of this auxiliary result (see Lemma 6.4) into four statements. We will use the following notation

$$\Omega_{\epsilon,x} = \{f \in C^\infty(T^{n-1}) : f = 0 \text{ on } U_\epsilon(x)\}, \quad 0 < \epsilon < 1.$$

Lemma 6.1. Let $n \geq 3$, and let $\lambda, x \in T^{n-1}$ be two orthogonal vectors. Assume that any $f \in \Omega_{\frac{1}{4},x}$ satisfies $\psi^{-1}(\lambda) = 0$. Then for any pair of orthogonal vectors

$u, v \in T^{n-1}$ we have $f \in \Omega_{\frac{1}{4},u}$ implies $\psi^{-1} f(v) = 0$.

Proof. For any two pairs of orthogonal unit vectors $(\lambda, x), (u, v)$ there exists a rotation $\rho \in TO(n)$ satisfying $u = \rho(x), v = \rho(\lambda)$. Since ψ^{-1} commutes with rotation, the result follows.

Lemma 6.2. Let $n \geq 3$, and let $\lambda \in x^{\perp}$. Assume that any $f \in \Omega_{\frac{1}{4},x}$ satisfies $\psi^{-1} f(\lambda) = 0$. Then $\psi^{-1}\left(\Omega_{\frac{1}{2},x}\right) \subset \Omega_{\frac{1}{4},\lambda}$

Proof. Take any $u \in U_{\frac{1}{4}}(\lambda)$.

Let $\rho \in TO(n),\ \rho(\lambda) = u,$

where λ is rotated into u inside $U_{\frac{1}{4}}(\lambda)$ in the plane containing λ, u and the origin. Then $\rho(x) \in U_{\frac{1}{4}}(x)$, and

$\Omega_{\frac{1}{2},x} \subset \Omega_{\frac{1}{4},\rho(x)}$. Moreover, $\psi^{-1} f(u) = 0$ since ψ^{-1} commutes with rotations. The point u was chosen arbitrarily in $U_{\frac{1}{4}}(\lambda)$, hence $\psi^{-1}\left(\Omega_{\frac{1}{2},x}\right) \subset \Omega_{\frac{1}{4},\lambda}$.

Lemma 6.3. Let $n \geq 3$, and let $\lambda \in x^{\perp}$. Then there exists a function $f = f_{x,\lambda}$ on T^{n-1} satisfying $f_{x,\lambda} = 0$ on $U_{\frac{1}{4}}(x)$, but $\psi^{-1} f_{x,\lambda}(\lambda) \neq 0$.

Proof. Assume the contrary. Then $\psi^{-1}\left(\Omega_{\frac{1}{2},x}\right) \subset \Omega_{\frac{1}{4},\lambda}$ by Lemma 6.2. Take any vector $y \in T^{n-1}$, and find a vector $q \in x^{\perp} \cap y^{\perp}$. Let $\rho \in TO(n)$ by such that $\rho(x) = x, \rho(\lambda) = q$. Observe that $f \in \Omega_{\epsilon,x}$ implies $f(\rho(.)) \in \Omega_{\epsilon,x}$. Since ψ^{-1} commutes with rotations, $\psi^{-1}\left(\Omega_{\frac{1}{2},x}\right) \subset \Omega_{\frac{1}{4},\lambda}$ yields $\psi^{-1}\left(\Omega_{\frac{1}{2},x}\right) \subset \Omega_{\frac{1}{4},q}$. Take two pairs of orthogonal vectors (x, q) and (q, y). By Lemma 6.1, we have

$\psi^{-2} f(y) = 0$. Thus $\psi^{-2} f \equiv 0$, a contradiction.

Lemma 6.4. Let $n \geq 3$. Then there exists an $\epsilon > 0$ and an absolute constant $c > 0$ such that for any $x, \lambda \in T^{n-1}$, there exists an even functional $f_{x,\lambda}$ satisfying $f_{x,\lambda} = 0$ on $U_{\epsilon}(x)$, and $\psi^{-1} f_{x,\lambda} \geq c$ on $U_{\epsilon}(\lambda)$.

Proof. We fix points x and λ, and provide an $\epsilon > 0$ and $c > 0$ depending on x, λ such that there is a function $f_{x,\lambda}$ satisfying $f_{x,\lambda} = 0$ on $U_{\epsilon}(x)$, and $\psi^{-1} f_{x,\lambda} \geq c > 0$ on $U_{\epsilon}(\lambda)$. Then we use the compactness argument to prove the statement of the lemma. Let $\lambda \notin x^{\perp}$. Then there exists an $\epsilon > 0$, such that $\lambda \notin E_{\epsilon}(x)$. For any function g the values of ψg on $U_{\epsilon}(x)$ depend only on the values of g on $E_{\epsilon}(x)$. Hence, we may consider an even

C^{∞} −function g such that $g(\pm\lambda) > 0$ and $g(v) = 0$, for $v \in E_{\epsilon}(x)$ and define $f_{x,\lambda} = \psi g(x)$.

Let $\lambda \in x^{\perp}$. Then, the previous lemma implies the existence of $= \epsilon(x,\lambda) = \frac{1}{8}$,

and a function $f = f_{x,\lambda}$ on T^{n-1} satisfying $f_{x,\lambda} = 0$ on $U_{\epsilon}(x)$, but $\psi^{-1} f_{x,\lambda}(\lambda) > 0$

(change the sign of $f_{x,\lambda}$ if necessary).

Thus, we proved that for any x and λ, there is $\epsilon' = \epsilon'(x,\lambda) > 0$

and there is a function $f_{x,\lambda}$ such that $f_{x,\lambda} = 0$ on $U_{\epsilon'}(x)$, but

$\psi^{-1} f_{x,\lambda}(\pm\lambda) \geq c'$, $c' = c'(x,\lambda) > 0$.

From the continuity of the function $\psi^{-1} f_{x,\lambda}$ we get that

$\psi^{-1} f_{x,\lambda} \geq c$, $c = c(x,\lambda) > 0$

on $U_{\epsilon''}(\lambda)$, for some $\epsilon'' > 0$.

Take $\tilde{\epsilon} = \tilde{\epsilon}(x,\lambda) = \min(\epsilon', \epsilon'')$.

We show that for any x and λ, there is

$\tilde{\epsilon} = \tilde{\epsilon}(x,\lambda) > 0$ and there is a function $f_{x,\lambda}$ such that $f_{x,\lambda} = 0$ on $U_{\tilde{\epsilon}}(x)$, but $\psi^{-1} f_{x,\lambda} \geq c$ on $U_{\tilde{\epsilon}}(\lambda), c = c(x,\lambda) > 0$.

Now we use the compactness argument to prove that we can choose an ϵ and c independent of x and λ. We choose a finite set of $\{x_i, \lambda_i\}_{i=1}^{m}$ such that

$\left\{U_{\frac{\tilde{\epsilon}_i}{2}}(x_i) \times U_{\frac{\tilde{\epsilon}_i}{2}}(\lambda_i)\right\}_{i=1}^{m}$ covers $T^{n-1} \times T^{n-1}$. We take

$\epsilon = \frac{1}{2} \min_{1 \leq i \leq m} \tilde{\epsilon}_i$ and $c = \min_{1 \leq i \leq m} c(x_i, \lambda_i)$.

Then for any (x,λ) there is a (x_i,λ_i) such that

$$U_{\epsilon}(x) \times U_{\epsilon}(\lambda) \subset U_{\tilde{\epsilon}_i}(x_i) \times U_{\tilde{\epsilon}_i}(\lambda_i)$$

and we may define $f_{x,\lambda} = f_{x_i,\lambda_i}$.

Theorem 6.5. Let $n \geq 5$. There exists a conver body K that is not an intersection body, such that $\forall \in xT^{n-1}$ there exists an $\epsilon(x)$ and an intersection body L_x such that $\rho_K = \rho_{L_x}$ on $U_{\epsilon(x)}(x)$.

Proof. Repeat the Proof of Theorem 1(see [22]).

Conclusion

Finally we have show that there is no local equatorial characterization of zonoids in odd dimensions. This gives a negative answer to the conjecture posed by W.Weil in 1977 and shows that the local equatorial characterization of zonoids may be given only in even dimensions. In addition we prove a similar result for intersection bodies and show that there is no local characterization of these bodies.

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