Application on Approximate Controllability of Semi-Linear Neutral Integro-Differential Equations with Nonlocal Conditions

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Abstract:

For validity we show an applicationon the approximate controllability of a semi-linear neutral integro-differential equation with nonlocal conditions by theory of analytic semi-groups, fractional powers of operators, and resolvent operator theory,considered by Hai Huang and Xianlong Fu1 [1]. Under the assumption of controllability on the corresponding linear system, we obtain the sufficient conditions for the considered semi-linear integro-differential system. In particular, the compactness condition or Lipschitz condition for the function g in the nonlocal condition appearing in literature is not required here. An example is also provided to illustrate the application of the obtained results.

Keywords: Neutral integro-differential equation, Approximate controllability, Resolvent operator, Fractional power operator, Nonlocal condition system.

1 Introduction

We show the approximate controllability of the following semi-linear neutral integro-differential equation with nonlocal conditionconsidered by Hai Huang and Xianlong Fu1 [1]:

$$\begin{cases} \dfrac{d}{d(1+\epsilon)}\left[x_r(1+\epsilon)+F\left(1+\epsilon, x_r\left(h_1(1+\epsilon)\right)\right)\right] = -Ax_r(1+\epsilon) \\ +(1+\epsilon)\displaystyle\int_{\gamma}^{0}(1+\epsilon-s)x_r(s)ds + G\left(1+\epsilon, x_r\left(h_2(1+\epsilon)\right)\right) + Bu(1+\epsilon), \qquad \epsilon \geq -1, \\ x_r(0)+g(x_r) = (x_r)_0, \end{cases} \tag{1}$$

where the state variable $x_r(\cdot)$ takes values in a Hilbert space X and the control function $u(\cdot)$ is given in the Banach space $L^2([0,1+2\epsilon];U)$ with U also a Hilbert space. The (unbounded) linear operator $-A$ generates an analytic semigroup on X. B is a bounded linear operator from U into $X, \gamma\ (1+\epsilon)$ is a family of closed linear operators, F, G, g, h_1 and h_2 are given functions to be specified later.

Grimmer et al. [2–3] proved the existence of solutions of the following integrodifferential evolution equation in Banach space X:

$$\begin{cases} v'(1+\epsilon) = Av(1+\epsilon) + \int_0^{1+\epsilon} \gamma(1+\epsilon-s)v(s)ds + f\,(1+\epsilon), & \epsilon \geq -1\,, \\ v(0) = v_0 \in X, \end{cases} \quad (2)$$

where $f: \mathbb{R}^+ \to X$ is a continuous function. The author(s) showed the representation, existence, and uniqueness of solutions of equation (2) via the resolvent operator associated to the following linear homogeneous equation:

$$\begin{cases} v'(1+\epsilon) = Av(1+\epsilon) + \int_0^{1+\epsilon} \gamma(1+\epsilon-s)v(s)ds, & \epsilon \geq -1\,, \\ v(0) = v_0 \in X. \end{cases}$$

The resolvent operator, replacing the role of C_0-semi-group for evolution equations, plays an important role in solving (2) in weak and strict senses. For example, in [4, 5, 6, 7], the authors have investigated the existence of solutions of this kind of equations, while the equations of stability and (almost) periodicity were explored respectively in [8, 9] and [10, 11].

As well known, controllability is one of the elementary concepts in mathematical control theory. Roughly speaking, controllability means that we can steer the state of dynamical system to an appropriate state by the control function existing in the equation (see [12,13]). The equation of controllability for various types of integro-differential systems has been extensively studied by mathematicians recently (we can refer to [14, 15, 16, 17]). Dieye et al. [18] studied the exact controllability for integro-differential equations in a real separable Hilbert space driven. Yang et al. [19] considered the exact controllability for a class of integro-differential evolution equations involving nonlocal initial conditions in Banach spaces. However, as stated in [20], the concept of exact controllability is very limited to infinite-dimensional systems. In [21], Bashirov and Mahmudov presented a so-called resolvent condition for the approximate controllability which is actually equivalent to the approximate controllability of linear equations. Since this condition is convenient to be applied, it is widely used to study the controllability equations for nonlinear controlled systems in infinite dimension (see [22, 23, 24, 25, 26]. Mokkedem and Fu [24] established the sufficient conditions for the approximate controllability of following semi-linear neutral integro-differential equations with finite delay:

$$\begin{cases} \dfrac{d}{d(1+\epsilon)}[x_r(1+\epsilon)+F\,(1+\epsilon,(x_r)_{1+\epsilon})] = -Ax_r(1+\epsilon)+\displaystyle\int_0^{1+\epsilon}\gamma\,(1+\epsilon-s)x_r(s)ds \\ \qquad +G(1+\epsilon,(x_r)_{1+\epsilon})+Bu(1+\epsilon), \qquad \epsilon\geq -1, \\ (x_r)_0=\phi, \qquad t\in[-r,0]. \end{cases}$$

The main techniques are utilizing the resolvent condition and fixed point principle.

Nonlocal initial condition is much better than the classical initial condition $x_r(0)=(x_r)_0$ when used to describe some physical phenomena. The nonlocal Cauchy equation of evolution equations was studied by Byszewski et al. [27] and the importance of nonlocal conditions in different fields has been discussed in [27], [28] and various papers (see [29, 30, 31, 32, 33, 34, 35, 36, 26–37]). Chang and Liu [29] proved the existence of mild and strong solutions for some neutral partial differential equations with nonlocal conditions in the α-norm. [35] considered the following semi-linear integro-differential equations with nonlocal Cauchy equation:

$$\begin{cases} u'(1+\epsilon)=A\left[x_r(1+\epsilon)+\displaystyle\int_0^{1+\epsilon}F(1+\epsilon-s)u(s)ds\right]+f\big(1+\epsilon,u(1+\epsilon)\big),\epsilon\geq -1, \\ u(0)+g\left(t_1,\cdots,t_p,u(t_1),\cdots,u\big(t_p\big)\right)=u_0. \end{cases} \tag{3}$$

There they studied the existence, uniqueness, and regularity of mild solutions of equation (3) using a fixed point principle. In [26], Vijayakumar discussed the approximate controllability for the following neutral integro-differential equations with nonlocal condition:

$$\begin{cases} \dfrac{d}{d(1+\epsilon)}\big[x_r(1+\epsilon)+g\big(1+\epsilon,x_r(1+\epsilon)\big)\big]=A\left[x_r(1+\epsilon)+\displaystyle\int_0^{1+\epsilon}G(1+\epsilon-s)x_r(s)ds\right] \\ \qquad +F\big(1+\epsilon,x_r(1+\epsilon)\big)+Bu(1+\epsilon), \qquad \epsilon\geq -1, \\ x_r(0)+h\big(x_r(1+\epsilon),x_r(1+2\epsilon),\cdots,x_r(1+p\epsilon)\big)=(x_r)_0, \end{cases} \tag{4}$$

where $h: I^p\times X\to X$ is a given function. The author obtained the approximate controllability result for equation. (4) under the assumptions that h satisfied Lipschitz condition and the corresponding linear system of equation. (4) is approximately controllable. In [26], Vijayakumar also studied the approximate controllability for a class of fractional differential inclusion with nonlocal conditions, where the linear part is nondensely defined and satisfies the Hille–Yosida condition.

Hai Huang and Xianlong Fu1 [1] discuss the approximate controllability of the integro-differential equation (1). A motivation of this paper is the approximate controllability equation of the following neutral partial integro-differential control equation:

$$\begin{cases} \dfrac{\partial}{\partial(1+\epsilon)}\left[z(1+\epsilon,x_r)+F\left(1+\epsilon,z(\cdot,x_r),\dfrac{\partial}{\partial x_r}z(\cdot,x_r)\right)\right]=\dfrac{\partial^2}{\partial x_r^2}z(1+\epsilon,x_r) \\ \qquad\qquad +\displaystyle\int_0^{1+\epsilon} b(1+\epsilon-s)\frac{\partial^2}{\partial x_r^2}z(s,x_r)ds \\ \qquad +G\left(1+\epsilon,z(\cdot,x_r),\dfrac{\partial}{\partial x_r}z(\cdot,x_r)\right)+Bu(1+\epsilon), \qquad 0\le x_r\le\pi, \qquad \epsilon\ge -1\,, \\ z(1+\epsilon,0)=z(1+\epsilon,\pi)=0, \qquad \epsilon\ge -1, \\ z(0,x_r)+g(z(1+\epsilon,x_r))=z_0(x_r), \qquad 0\le x_r\le\pi. \end{cases} \tag{5}$$

This system arises in the study of heat flow in materials with nonlocal condition; here $z(1+\epsilon,x_r)$ represents the temperature of a conduct of the pointsx_r and time t. It can be rewritten as the abstract form Eq. 1. Another motivation is to develop the work in [24] to the systems with nonlocal conditions.

We shall first show the existence of mild solutions of equation (1) for different control functions, and discuss the approximate controllability equation applying the resolvent condition. We investigate the existence equation by restricting the mild solutions in a Hilbert space $X_{1+\epsilon}(\subset X)$ induced by fractional power of operator A and adopt the arguments on fractional powers of operators. We study the approximate controllability for Eq. 1 in the original space X.

The obtained results have more general applications in subspace $X_{1+\epsilon}$ and as a result they are still valid for the PDE systems with the nonlinear terms involving spatial derivatives. Clearly, the obtained results in this paper extend straight to the works in [38, 16, 26] and develop the corresponding results in [24, 25].

We recall some concepts, hypotheses, and basic results about resolvent operator and facts to be used. We focus our attention on the existence of mild solutions for system (1), we discuss the approximate controllability for this system and obtain our main results. An example is provided to illustrate the applications of the approximate controllability theorem (see [1]).

2 Preliminaries

For$X$ be a Hilbert space and the operator $-A: D(A)\subseteq X\to X$ be the infinitesimal generator of a compact analytic semigroup $\left(S(1+\epsilon)\right)_{\epsilon\ge -1}$. Y is the Banach space formed from $D(A)$ with the graph norm $\|y_r\|_Y=\|Ay_r\|+\|y_r\|$, for$y_r\in D(A)$. Let $0\in\rho(A)$, the resolvent set of operator A, for $-1<\epsilon<0$, we can define the fractional power $A^{1+\epsilon}$, as a closed linear operator on its domain $D(A^{1+\epsilon})$. Furthermore, the subspace $D(A^{1+\epsilon})$ is dense in X and the expression

$$\|x_r\|_{1+\epsilon}=\|A^{1+\epsilon}x_r\|, \qquad x_r\in D(A^{1+\epsilon}),$$

defines a norm on $D(A^{1+\epsilon})$. Denote the space $(D(A^{1+\epsilon}), \|\cdot\|_{1+\epsilon})$ by $X_{1+\epsilon}$, then it is well known that for each $-1 < \epsilon < 0, X_{1+\epsilon}$ is a Banach space. Moreover, $X_{1+2\epsilon} \hookrightarrow X_{1+\epsilon}$ for $-1 < \epsilon < 0$and the imbedding is compact whenever $R(\lambda, A)$, the resolvent operator of A, is compact. Hereafter, we denote by $C([0, 1+2\epsilon], X_{1+2\epsilon})$ the Banach space of continuous functions from $[0, 1+2\epsilon]$ to $X_{1+2\epsilon}$ with the norm:

$$\|x_r\|_C = \sup_{\epsilon \geq -1} \|A^{1+2\epsilon} x_r(1+\epsilon)\|, \qquad x_r \in C([0, 1+2\epsilon], X_{1+2\epsilon}).$$

For the theory of operator semi-group, see [39] and [40].

The theory of resolvent operator plays an essential role in investigating the existence of solutions of Eq. 1. We collect the definition and some basic results about this theory see [2–3].

Definition 2.1 (see [2]) A family of bounded linear operators $W(1+\epsilon) \in \mathscr{L}(X)$ for $\epsilon \geq -1$ is called resolvent operators for:

$$\begin{cases} \dfrac{d}{d(1+\epsilon)} x_r(1+\epsilon) = -Ax_r(1+\epsilon) + \displaystyle\int_0^{1+\epsilon} \gamma(1+\epsilon-s)x_r(s)ds, \qquad \epsilon \geq 0, \\ x_r(0) = (x_r)_0 \in X, \end{cases} \tag{6}$$

if

(i) $W(0) = I$ and $\|W(1+\epsilon)\| \leq N_1 e^{\omega(1+\epsilon)}$ for some $N_1 > 0, \omega \in \mathbb{R}$.

(ii) For all $x_r \in X, W(1+\epsilon)x_r$ is continuous for $\epsilon \geq -1$.

(iii)$W(1+\epsilon) \in \mathcal{L}(Y)$, for $\epsilon \geq -1$. For $x_r \in Y$, $W(1+\epsilon)x_r \in C^1([0, 1+2\epsilon], X) \cap C([0, 1+2\epsilon], Y)$ and for $\epsilon \geq -1$ such that

$$W'(1+\epsilon)x_r = -AW(1+\epsilon)x_r + \int_0^{1+\epsilon} \gamma(1+\epsilon-s)W(s)x_r ds$$
$$= -W(1+\epsilon)Ax_r + \int_0^{1+\epsilon} W(1+\epsilon-s)\gamma(s)x_r ds.$$

Suppose that A and $\gamma(\cdot)$ in Eq. (6) satisfy the hypotheses (V_1)– (V_3) in [20]. Then, it follows from [3] that, under these conditions, there is a family of resolvent operator $\big(W(1+\epsilon)\big)_{\epsilon \geq -1}$ for the linear Eq (6) defined by

$$W(0) = I$$

And

$$W(1+\epsilon)x_r = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda(1+\epsilon)}\left(\lambda I - A - \gamma^*(\lambda)\right)^{-1} x_r d\lambda = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda(1+\epsilon)}\rho(\lambda)x_r d\lambda, \qquad \epsilon \geq 0,$$

WhereΓ is a contour of the type used to obtain an analytic semigroup and $\gamma^*(\lambda)$ denotes the Laplace transform of $\gamma(1+\epsilon)$. We can select contour to be included in the region

$\Lambda = \left\{\lambda \in C: |\arg\lambda| < \frac{\pi}{2} + \delta, 0 < \delta < \frac{\pi}{2}\right\}$, consisting of Γ_1, Γ_2, and Γ_3, where

$$\Gamma_1 = \left\{re^{i\phi}: r \geq 1\right\}, \qquad \Gamma_3 = \left\{re^{-i\phi}: r \geq 1\right\}, \qquad \frac{\pi}{2} < \phi < \frac{\pi}{2} + \delta,$$

$$\Gamma_2 = \left\{e^{i\theta}: -\phi \leq \theta \leq \phi\right\},$$

oriented so that $Im(\lambda)$ is increasing on Γ_1 and Γ_2. Moreover, $W(1+\epsilon)$ is also analytic and there exist $N, C_{1+\epsilon} > 0$ such that:

$$\|W(1+\epsilon)\| \leq N \quad \text{and} \quad \|A^{1+\epsilon}W(1+\epsilon)\| \leq \frac{C_{1+\epsilon}}{(1+\epsilon)_{1+\epsilon}}, \qquad -1 \leq \epsilon < 0. \quad (7)$$

For the resolvent operator $W(1+\epsilon)$, we have:

Lemma 2.1 (see [31, 32]) $W(1+\epsilon)$ and $AW(1+\epsilon)$ are both continuous for $\epsilon \geq 0$ in the uniform operator topology of (X).

We will also assume that:

$$A^{1+\epsilon}W(1+\epsilon)x_r = W(1+\epsilon)A^{1+\epsilon}x_r, \qquad \text{for any} \quad -1 < \epsilon < 0, \qquad x_r \in D(A^{1+\epsilon}). \qquad (8)$$

It is easy to see that this commutation is not always valid. We point out, however, that this property can be reached in many cases. For example, let $\gamma(1+\epsilon-s) = b(1+\epsilon-s)A$ with b(t) a scalar function defined on $(0, +\infty)$, then the linear Eq (6) becomes:

$$\begin{cases} \dfrac{d}{d(1+\epsilon)}x_r(1+\epsilon) = -Ax_r(1+\epsilon) + \displaystyle\int_0^{1+\epsilon} b(1+\epsilon-s)Ax_r(s)ds, \\ x_r(0) = (x_r)_0 \in X. \end{cases} \qquad (9)$$

If we impose the following conditions on the equation equation (9),

(V_1') A generates an analytic semi-group on X. In particular,

$$\Lambda 1 = \left\{\lambda \in \mathbb{C}: |\arg\lambda| < \left(\frac{\pi}{2}\right) + \delta_1\right\}, 0 < \delta_1 < \pi/2$$

is contained in the resolvent set of A and $\|(\lambda I - A)^{-1}\| \leq M/|\lambda|$ on Λ_1 for some constant $M > 0$. The scalar function $b(\cdot)$ is in $L^1(0,\infty)$ with $b^*(\lambda)$ absolutely convergent for $Re\lambda > 0$, where $b^*(\lambda)$ denotes the Laplace transform of $b(1+\epsilon)$.

(V_2') There exists $\Lambda = \left\{\lambda \in \mathbb{C}: |\arg\lambda| < \left(\frac{\pi}{2}\right) + \delta_2\right\}, 0 < \delta_2 < \pi/2$, so that $\lambda \in \Lambda$ implies $\psi(\lambda) = 1 + b^*(\lambda)$ exists and is not zero. Further, $\lambda\psi^{-1}(\lambda) \in \Lambda_1$ for $\lambda \in \Lambda$.

$(V_3')Ir\ \Lambda, b^*(\lambda) \to 0$ as $|\lambda| \to \infty$.

Then, from [3], the conditions (V_1)–(V_3) there are fulfilled and hence the resolvent operator $W(1+\epsilon)$ is analytic. It is easy to see that Eq. 8 holds in this situation.

We define the mild solutions of Eq. 1 expressed by the resolvent operator $W(1+\epsilon)$ as follows (see [1]).

Definition 2.2 A function $x_r(\cdot) \in C([0, 1+2\epsilon], X_{1+\epsilon})$ is said to be a mild solution of Eq. 1 if:

$$x_r(1+\epsilon) = W(1+\epsilon)\left[(x_r)_0 + F\left(0, x_r\left(h_1(0)\right)\right) - g(x_r)\right] - F\left(1+\epsilon, x_r\left(h_1(1+\epsilon)\right)\right)$$
$$+ \int_0^{1+\epsilon} AW(1+\epsilon-s)F\left(s, x_r\left(h_1(s)\right)\right)ds$$
$$- \int_0^{1+\epsilon} W(1+\epsilon-s)\int_0^s \gamma(s-\tau)F\left(\tau, x_r\left(h_1(\tau)\right)\right)d\tau\, ds$$
$$+ \int_0^{1+\epsilon} W(1+\epsilon-s)\left(G\left(s, x_r\left(h_2(s)\right)\right) + Bu(s)\right)ds,$$

for $\epsilon \geq -1$.

We give present the concepts of approximate controllability.

Definition 2.3 equation (1) is said to be approximately controllable on the interval $I = [0, 1+2\epsilon]$ if $\mathcal{R}(1+2\epsilon, (x_r)_0)$ is dense in X, i.e.:

$$\overline{\mathcal{R}(1+2\epsilon, (x_r)_0)} = X,$$

where $\mathcal{R}(1+2\epsilon, (x_r)_0) = \{x_r(1+2\epsilon, (x_r)_0, u), u(\cdot) \in L^2(I;U)\}$.

As mentioned in Section 1, we shall discuss the approximate controllability for system (1) by applying the so-called resolvent condition introduced in [21]. For this, we need to define the following operators.

$$\Gamma_{1+2\epsilon} = \Gamma \int_0^{1+2\epsilon} W(1+2\epsilon-s)BB^*W^*(1+2\epsilon-s)ds,$$

$$R(\lambda, \Gamma_{1+2\epsilon}) = (\lambda I + \Gamma_{1+2\epsilon})^{-1},$$

where B^* and $W^*(1+\epsilon)$ denote the adjoints of the operators B and $W(1+\epsilon)$, respectively. Since $\Gamma_{1+2\epsilon}: X \to X$ is clearly positive, we see $R(\lambda, \Gamma_{1+2\epsilon})$ is well defined on X. In the sequel, we always assume that the operator $R(\lambda, \Gamma_{1+2\epsilon})$ satisfies that:

(H_0) $\lambda R(\lambda, g_{1+2\epsilon}) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

It is not difficult to prove by using the similar arguments as in [21] that:

Theorem 2.1 Hypothesis (H_0) holds if and only if the linear control system corresponding to system (1):

$$\begin{cases} \dfrac{d}{d(1+\epsilon)} x_r(1+\epsilon) = -Ax_r(1+\epsilon) + \displaystyle\int_0^{1+\epsilon} \gamma(1+\epsilon-s)x_r(s)ds + Bu(1+\epsilon), & \epsilon \geq -1, \\ x_r(0) = (x_r)_0, \end{cases}$$

is approximately controllable on $[0, 1+2\epsilon]$.

We end this section by stating the following fixed point principle.

Theorem 2.2 (see [41]) Assume that P is a condensing operator on a Banach space X, i.e., P is continuous and takes bounded sets into bounded sets, and $(1+\epsilon)\big(P(B)\big) \leq (1+\epsilon)(B)$ for every bounded set B of X with $(1+\epsilon)(B) > 0$. If $P(H) \subseteq H$ for a convex, closed, and bounded set H of X, then P has a fixed point in H (where $(1+\epsilon)(\cdot)$ denotes the Kuratowski measure of noncompactness).

3 Approximate Controllability

We study the approximate controllability of System (1). We first prove the existence of solutions of System (1). We then show that under certain assumptions, the approximate controllability of Eq. 1 is implied by the approximate controllability of the corresponding linear system. That is, for any $x_r^{1+2\epsilon} \in X$, by choosing proper control u^λ (for any given $\lambda \in (0,1)$), we show there is a mild solution

$x_r^\lambda(0, (x_r)_0, u) \in C([0, 1+2\epsilon];\ X^{1+\epsilon})$ of System (1), then we use (H_0) to prove that

$x_r^\lambda(1+2\epsilon) \to x_r^{1+2\epsilon}$ in X as $\lambda \to 0^+$ which means that Eq. 1 is approximately controllable on

$[0, 1+2\epsilon]$.

To follow, we make the following restrictions on Eq. 1. Let $-1 < \epsilon < 0$.

$(H_1) W(1+\epsilon)$ is a compact operator for each $\epsilon \geq 0$.

$(H_2) B \in \mathfrak{L}(U; X)$, i.e., B is a bounded linear operator from U to X. Let $\|B\| = M$.

(H_3) There exists a constant $-1 < \epsilon < 0$, such that $F: [0, 1+2\epsilon] \times X_{1+2\epsilon} \to X_{2+3\epsilon}$ satisfies the Lipschitz condition, i.e., there exists a constant $\epsilon \geq 0$ such that:

$$\|F(1+\epsilon, (x_r)_1) - F(1+2\epsilon, (x_r)_2)\|_{2+3\epsilon} \leq (1+\epsilon)(|\epsilon| + \|(x_r)_1 - (x_r)_2\|_{1+2\epsilon})$$

for any $\epsilon \geq -1, (x_r)_1, (x_r)_2 \in X_{1+2\epsilon}$, and the inequality

$$\|F(1+\epsilon, x_r)\|_{2+3\epsilon} \leq (1+\epsilon)(\|x_r\|_{1+2\epsilon} + 1)$$

holds for any $(1+\epsilon, x_r) \in [0, 1+3\epsilon] \times X_{1+2\epsilon}$.

$(H_4) \{\gamma(1+\epsilon)\}_{\epsilon \geq -1}$ is a family of operators from Y to X such that $\gamma(1+\epsilon) \in \mathscr{L}(X_{2+3\epsilon}, X)$ for each $\epsilon \geq -1$. Then, there exists a positive number M_1 such that:

$$\|\gamma(1+\epsilon)\|_{2+3\epsilon}, \qquad 0 \leq M_1, \qquad \epsilon \geq -1.$$

(H_5) The function $G: [0, 1+3\epsilon] \times X_{1+2\epsilon} \to X$ verifies the following conditions:

(i) For each $\epsilon \geq -1$, the function $G(1+\epsilon, \cdot): X_{1+2\epsilon} \to X$ is continuous and for each $x_r \in X_{1+2\epsilon}$ the function $G(\cdot, x_r): [0, 1+2\epsilon] \to X$ is strongly measurable;

(ii) For each positive number $k \in \mathbb{N}$, there is a positive function $g_k \in L^\infty([0, 1+2\epsilon])$ such that:

$$\sup_{x_r \leq k} \|G(1+\epsilon, x_r)\| \leq g_k(1+\epsilon).$$

$(H_6) h_i \in C([0, 1+2\epsilon]; [0, 1+2\epsilon]), i = 1, 2$. And the function $g: C([0, 1+2\epsilon], X_{1+2\epsilon}) \to X_{1+2\epsilon}$ is a continuous mapping which maps bounded sets into bounded sets, i.e., there exists a constant $\epsilon \geq 0$ such that for any $x_r \in C([0, 1+2\epsilon], X_{1+2\epsilon})$,

$$\|g(x_r)\|_{1+2\epsilon} \leq (1+\epsilon)\|x_r\|_C.$$

Moreover, there is a $s + \epsilon = \delta(k) \in (0, s+2\epsilon)$ such that $g(x_r) = g(y_r)$ for any $x_r, y_r \in B_k$ with $x_r(s) = y_r(s), \epsilon \geq 0$, where $B_k = \{x_r \in C([0, s+2\epsilon], X_{1+2\epsilon}), \|x_r(\cdot)\|_C \leq k\}$.

For any given $x_r^{1+2\epsilon} \in X, \lambda \in (0, 1)$ and for any $x_r(\cdot) \in C([0, 1+2\epsilon], X_{1+2\epsilon})$, we take the control function $u^\lambda(1+\epsilon)$, simply denoted by $u(1+\epsilon)$ as:

$$u(1+\epsilon) = B^*W^*(\epsilon)R(\lambda,\Gamma_{1+2\epsilon})\Big[x_r^{1+2\epsilon} - W(1+2\epsilon)\Big((x_r)_0 + F\Big(0, x_r\big(h_1(0)\big)\Big) - g(x_r)\Big)$$
$$+ F\Big(1+2\epsilon, x_r\big(h_1(1+2\epsilon)\big)\Big) - \int_0^{1+2\epsilon} AW\,(1+2\epsilon-s)F\Big(s, x_r\big(h_1(s)\big)\Big)\,ds$$
$$+ \int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^s \gamma(s-\tau)F\Big(\tau, x_{rr}\big(h_1(\tau)\big)\Big)\,d\tau\;ds$$
$$- \int_0^{1+2\epsilon} W(1+2\epsilon$$
$$- s)G\Big(s, x_r\big(h_2(s)\big)\Big)\,ds\Big]. \qquad (10)$$

Using this control, we define the operator Q^λ on $C([0,1+2\epsilon];\ X_{1+2\epsilon})$ as, for $\epsilon \geq -1$:

$$\big(Q^\lambda x_r\big)(1+\epsilon)$$
$$= W(1+\epsilon)\Big((x_r)_0 + F\Big(0, x_r\big(h_1(0)\big)\Big) - g(x_r)\Big) - F\Big(1+\epsilon, x_r\big(h_1(1+\epsilon)\big)\Big)$$
$$+ \int_0^{1+\epsilon} AW\,(1+\epsilon-s)F\Big(s, x_r\big(h_1(s)\big)\Big)\,ds$$
$$- \int_0^{1+\epsilon} W(1+\epsilon-s)\int_0^s \gamma(s-\tau)F\Big(\tau, x_r\big(h_1(\tau)\big)\Big)\,d\tau\,ds$$
$$+ \int_0^{1+\epsilon} W(1+\epsilon-s)\Big(G\Big(s, \big(h_2(s)\big)\Big) + Bu(s)\Big)\,ds.$$

We now establish the existence result firstly (see [1]).

Theorem 3.1 Assume that the assumptions (H_0)–(H_6) are satisfied. Then, the equation (1) has at least one mild solution on $[0,1+2\epsilon]$ provided that:

$$K_0 := \Big[(N+1)\big\|A^{-(1+\epsilon)}\big\| + \frac{(1+2\epsilon)^{1+\epsilon}C_{-\epsilon}}{1+\epsilon} - \frac{(1+2\epsilon)^{-2\epsilon}C_{1+2\epsilon}M_1}{2\epsilon} + \frac{(1+2\epsilon)^{-2\epsilon}C_{1+2\epsilon}M^2N}{\lambda(-2\epsilon)}$$
$$\times\left((N+1)\big\|A^{-(2+3\epsilon)}\big\| + \frac{C_{-(1+3\epsilon)}(1+2\epsilon)^{2+3\epsilon}}{2+3\epsilon} + NM_1(1+2\epsilon)^2\right)\Big](1+\epsilon) + N^2$$
$$+ \frac{(1+2\epsilon)^{-2\epsilon}C_{1+2\epsilon}M^2N}{\lambda(-2\epsilon)}\big\|A^{-(1+2\epsilon)}\big\|_{1+\epsilon} < 1. \qquad (11)$$

As we pointed out in Section 1, because the function g does not satisfy the Lipschitz or compactness condition, we can not prove this theorem by directly applying Theorem 2.2. Hence, we will show this

result by using the approximation arguments. For this purpose, we establish two auxiliary lemmas. For a fixed $r \in \mathbb{N}$, let us consider the following approximate problem:

$$\begin{cases} \dfrac{d}{d(1+\epsilon)}\left[x_r(1+\epsilon) + F\left(1+\epsilon, x_r\left(h_1(1+\epsilon)\right)\right)\right] \\ = -Ax_r(1+\epsilon) + 1 + \epsilon \displaystyle\int_{\gamma}^{0} (1+\epsilon-s)x_r(s)ds + G\left(1+\epsilon, x_r\left(h_2(1+\epsilon)\right)\right) + Bu(1+\epsilon), \\ \epsilon \geq -1, \qquad x_r(0) + W\left(\dfrac{1}{n}\right) g(x_r) = (x_r)_0 \end{cases} \tag{12}$$

Then, we have the following result (see [1]).

Lemma 3.1 Assume that all the conditions in Theorem 3.1 are satisfied, then for any $n \in \mathbb{N}$, the nonlocal Cauchy equation (12) has at least one mild solution $x_r(\cdot) \in C([0, 1+2\epsilon], X_{1+2\epsilon})$.

Proof. Let B_k be that defined in (H_6). It is obvious that B_k is a bounded, closed, and convex set in $C([0, 1+2\epsilon], X_{1+2\epsilon})$. In view of Eq. 10, we estimate the norm of the control function corresponding to the equation (12), that is,

$$\begin{aligned} \|u_n(s)\| = \Big\| & B^*W^*(1+2\epsilon-s)R(\lambda, \Gamma_{1+2\epsilon})\Big[x_r^{1+2\epsilon} \\ & - W(1+2\epsilon)\left((x_r)_0 + F\left(0, x_r\left(h_1(0)\right)\right) - W\left(\frac{1}{n}\right)g(x_r)\right) \\ & + F\left(1+2\epsilon, x_r\left(h_1(1+2\epsilon)\right)\right) - \int_0^{1+2\epsilon} AW(1+2\epsilon-s)F\left(s, x_r\left(h_1(s)\right)\right)ds \\ & + \int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^{s} \gamma(s-\tau)F\left(\tau, x_r\left(h_1(\tau)\right)\right)d\tau\, ds \\ & - \int_0^{1+2\epsilon} W(1+2\epsilon-s)G\left(s, x_r\left(h_2(s)\right)\right)ds\Big]\Big\| \end{aligned}$$

$$= \left\| B^*W^*(\epsilon)R(\lambda,\Gamma_{1+2\epsilon})\left[x_r^{1+2\epsilon}\right.\right.$$

$$- W(1+2\epsilon)\left((x_r)_0 + A^{-(2+3\epsilon)A^{2+3\epsilon}F\left(0,x_r(h_1(0))\right)} - A^{-(1+2\epsilon)}W\left(\frac{1}{n}\right)A^{1+2\epsilon}g(x_r)\right)$$

$$+ A^{-(2+3\epsilon)}A^{2+3\epsilon}F\left(1+2\epsilon, x_r(h_1(1+2\epsilon))\right)$$

$$- \int_0^{1+2\epsilon} A^{-(1+3\epsilon)}W(1+2\epsilon-s)A^{2+3\epsilon}F\left(s, x_r(h_1(s))\right)ds$$

$$+ \int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^s \gamma(s-\tau)F\left(\tau, x_r(h_1(\tau))\right)d\tau ds$$

$$\left.\left. - \int_0^{1+2\epsilon} W(1+2\epsilon-s)G\left(s, x_r(h_2(s))\right)ds\right]\right\|$$

$$\leq MN\frac{1}{\lambda}\left[\|x_r^{1+2\epsilon}\| + N\left(\|(x_r)_0\| + \left\|A^{-(2+3\epsilon)}\right\|(1+\epsilon)(k+1) + \left\|A^{-(1+2\epsilon)}\right\|N(1+\epsilon)k\right)\right.$$

$$+ \left\|A^{-(2+3\epsilon)}\right\|(1+\epsilon)(k+1) + \frac{C_{-(1+3\epsilon)}(1+2\epsilon)^{2+3\epsilon}}{2+3\epsilon}(1+\epsilon)(k+1)$$

$$\left. + NM_1(1+2\epsilon)^2(1+\epsilon)(k+1) + N\int_0^{1+2\epsilon} g_k(s)ds\right]$$

$$= MN\frac{1}{\lambda}\left[\|x_r^{1+2\epsilon}\| + N\|(x_r)_0\| + \left((N+1)\left\|A^{-(2+3\epsilon)}\right\| + \frac{C_{-(1+3\epsilon)}(1+2\epsilon)^{2+3\epsilon}}{2+3\epsilon} + NM_1(1+2\epsilon)^2\right)(1\right.$$

$$+ \epsilon)(k+1)\left\|A^{-(1+2\epsilon)}\right\|N^2(1+\epsilon)k + N(1$$

$$\left. + 2\epsilon)\,\|g_k(\cdot)\|_\infty\right]. \qquad (13)$$

Here, due to (H_0), we have used the fact w. l. o. g. that

$$\|R(\lambda,\Gamma_{1+2\epsilon})\| \leq \frac{1}{\lambda}, \qquad \text{for all} \quad \lambda \in (0,1).$$

Define now the operator Q_n on $C([0,1+2\epsilon],X_{1+2\epsilon})$ by the formula:

$$(Q_n x_r)(1+\epsilon)$$
$$= W(1+\epsilon)\left((x_r)_0 + F\left(0, x_r\left(h_1(0)\right)\right) - W\left(\frac{1}{n}\right)g(x_r)\right) - F\left(1+\epsilon, x_r\left(h_1(1+\epsilon)\right)\right)$$
$$+ \int_0^{1+\epsilon} AW(1+\epsilon-s)F\left(s, x_r\left(h_1(s)\right)\right)ds$$
$$- \int_0^{1+\epsilon} W(1+\epsilon-s)\int_0^s \gamma(s-\tau)F\left(\tau, x_r\left(h_1(\tau)\right)\right)d\tau\, ds$$
$$+ \int_0^{1+\epsilon} W(1+\epsilon-s)\left(G\left(s, x_r\left(h_2(s)\right)\right) + Bu_n(s)\right)ds.$$

Then, it is easy to see that a fixed point of Q_n is exactly a mild solution of Eq. 12. Subsequently, we will prove that Q_n has a fixed point on some B_k by applying Theorem 2.2.

We first claim that there exists $a\ k \in \mathbb{N}$ such that $Q_n(B_k) \subseteq B_k$, if it is not true, then for each $k \in \mathbb{N}$, there is a function $(x_r)_k(\cdot) \in B_k$, but $Q_n(x_r)_k \notin B_k$, that is $\|Q_n(x_r)_k(1+\epsilon)\|_{1+2\epsilon} > k$ for some

$t(k) \in [0, 1+2\epsilon]$, where $t(k)$ denotes t is dependent on k. On the other hand, however, we have by virtue of Eqs. 7 and 13 that:

$$k < \|(Q_n(x_r)_k)(1+\epsilon)\|_{1+2\epsilon}$$
$$\leq \left\|W(1+\epsilon)\left[(x_r)_0 + F\left(0, (x_r)_k\left(h_1(0)\right)\right) - W\left(\frac{1}{n}\right)g((x_r)_k)\right]\right\|_{1+2\epsilon}$$
$$+ \left\|F\left(1+\epsilon, (x_r)_k\left(h_1(1+\epsilon)\right)\right)\right\|_{1+2\epsilon}$$
$$+ \int_0^{1+\epsilon} \left\|A^{-\epsilon}W(1+\epsilon-s)A^{1+\epsilon}F\left(s, (x_r)_k\left(h_1(s)\right)\right)\right\|_{1+2\epsilon} ds$$
$$+ \int_0^{1+\epsilon} \|A^{1+2\epsilon}W(1+\epsilon-s)\| \int_0^s \left\|\gamma(s-\tau)F\left(\tau, x_r\left(h_1(\tau)\right)\right)\right\| d\tau ds$$
$$+ \int_0^{1+\epsilon} A^{1+2\epsilon}W(1+\epsilon-s)G\left(s, x_r\left(h_2(s)\right)\right)ds$$
$$+ \int_0^{1+\epsilon} \|A^{1+2\epsilon}W(1+\epsilon-s)Bu_n(s)\|ds$$

$$\leq N\left[\|(x_r)_0\|_{1+2\epsilon} + \left\|A^{-(1+\epsilon)}\right\|(1+\epsilon)(k+1) + N(1+\epsilon)k\right] + \left\|A^{-(1+\epsilon)}\right\|(1+\epsilon)(k+1)$$
$$+ \frac{(1+2\epsilon)^{1+\epsilon}C_{-\epsilon}}{1+\epsilon}(1+\epsilon)(k+1) - \frac{(1+2\epsilon)^{1-2\epsilon}C_{1+2\epsilon}M_1}{2\epsilon}(1+\epsilon)(k+1)$$
$$- \frac{(1+2\epsilon)^{-2\epsilon}C_{1+2\epsilon}}{2\epsilon}\|g_k(\cdot)\|_\infty$$

$$+\frac{(1+2\epsilon)^{-2\epsilon}C_{1+2\epsilon}M_2N}{\lambda(-2\epsilon)}[\|x_r^{1+2\epsilon}\|+N\|(x_r)_0\|$$
$$+\left((N+1)\left\|A^{-(2+3\epsilon)}\right\|+\frac{C_{1-(2+3\epsilon)}(1+2\epsilon)^{2+3\epsilon}}{2+3\epsilon}+NM_1(1+2\epsilon)^2\right)(1+\epsilon)(k+1)$$
$$+\left\|A^{-(1+2\epsilon)}\right\|N^2(1+\epsilon)k+N(1+2\epsilon)\|g_k(\cdot)\|_\infty.$$

Thus, dividing on both sides by the k and taking the lower limit as $k\to+\infty$, we then get:

$$\left[(N+1)\left\|A^{-(1+\epsilon)}\right\|+\frac{(1+2\epsilon)^{1+\epsilon}C_{-\epsilon}}{1+\epsilon}-\frac{(1+2\epsilon)^{1-2\epsilon}C_{1+2\epsilon}M_1}{2\epsilon}+\frac{(1+2\epsilon)^{-2\epsilon}C_{1+2\epsilon}M^2N}{\lambda(-2\epsilon)}\right.$$
$$\left.\times\left((N+1)\left\|A^{-(2+3\epsilon)}\right\|+\frac{C_{1-(2+3\epsilon)}(1+2\epsilon)^{2+3\epsilon}}{2+3\epsilon}+NM_1(1+2\epsilon)^2\right)\right](1+\epsilon)$$
$$+N^2\left(1+\frac{(1+2\epsilon)^{-2\epsilon}C_{1+2\epsilon}M^2N}{\lambda(-2\epsilon)}\left\|A^{-(1+2\epsilon)}\right\|\right),\qquad \epsilon\geq 0,$$

which contradicts (11). Hence, there is a positive integer k such that $Q_n(B_k)\subseteq B_k$.

We then decompose Q_n into two operators Q_{n1}, Q_{n2} on B_k as:

$$(Q_{n1}x_r)(1+\epsilon)$$
$$=W(1+\epsilon)F\left(0,x_r\left(h_1(0)\right)\right)-F\left(1+\epsilon,x_r\left(h_1(1+\epsilon)\right)\right)$$
$$+\int_0^{1+\epsilon}AW(1+\epsilon-s)F\left(s,x_r\left(h_1(s)\right)\right)ds,$$

$$(Q_{n2}x_r)(1+\epsilon)$$
$$=W(1+\epsilon)\left[(x_r)_0-W\left(\frac{1}{n}\right)g(x_r)\right]$$
$$-\int_0^{1+\epsilon}W(1+\epsilon-s)\int_0^s\gamma(s-\tau)F\left(\tau,x_r\left(h_1(\tau)\right)\right)d\tau\,ds$$
$$+\int_0^{1+\epsilon}W(1+\epsilon-s)\left(G\left(s,x_r\left(h_2(s)\right)\right)+Bu_n(s)\right)ds,$$

for all $\epsilon\geq -1$. Next, we will verify that Q_{n1} is a contraction while Q_{n2} is a compact operator.

To prove that Q_{n1} is a contraction, we take $(x_r)_1, (x_r)_2\in B_k$, then, for each $\epsilon\geq -1$, we find:

$$\|(Q_{n1}(x_r)_1)(1+\epsilon)-(Q_{n1}(x_r)_2)(1+\epsilon)\|_{1+2\epsilon}$$
$$\leq \left\|W(1+\epsilon)A^{-(1+\epsilon)}\left[A^{1+\epsilon}F\left(0,(x_r)_1(h_1(0))\right)-A^{1+\epsilon}F\left(0,(x_r)_2(h_1(0))\right)\right]\right\|_{1+2\epsilon}$$
$$+\|A^{-(1+\epsilon)}\|A^{1+\epsilon}F\left(1+\epsilon,(x_r)_1(h_1(1+\epsilon))\right)$$
$$-\left\|A^{1+\epsilon}F\left(1+\epsilon,(x_r)_2(h_1(1+\epsilon))\right)\right\|_{1+2\epsilon}$$
$$+\left\|\int_0^{1+\epsilon} A^{-\epsilon}W(1+\epsilon-s)A^{1+\epsilon}\left[F\left(s,(x_r)_1(h_1(s))\right)\right.\right.$$
$$\left.\left.-F\left(s,(x_r)_2(h_1(s))\right)\right]ds\right\|_{1+2\epsilon}$$

$$\leq N\|A^{-(1+\epsilon)}\|(1+\epsilon)\sup_{\epsilon\geq -1}\|(x_r)_1(1+\epsilon)-(x_r)_2(1+\epsilon)\|_{1+2\epsilon}+\|A^{-(1+\epsilon)}\|(1$$
$$+\epsilon)\sup_{\epsilon\geq -1}\|(x_r)_1(1+\epsilon)-(x_r)_2(1+\epsilon)\|_{1+2\epsilon}$$
$$+\int_0^{1+\epsilon}\frac{C_{-\epsilon}(1+\epsilon)}{(1+\epsilon-s)^{-\epsilon}}\|(x_r)_1(s)-(x_r)_2(s)\|_{1+2\epsilon}ds$$
$$\leq\left((N+1)A^{-(1+\epsilon)}+\frac{(1+2\epsilon)^{1+\epsilon}C_{-\epsilon}}{1+\epsilon}\right)(1$$
$$+\epsilon)\sup_{\epsilon\geq -1}\|(x_r)_1(1+\epsilon)-(x_r)_2(1+\epsilon)\|_{1+2\epsilon}.$$

Thus, we obtain

$$\|(Q_{n1}(x_r)_1)-(Q_{n1}(x_r)_2)\|_C\leq\left(NM_0+M_0+\frac{C_{-\epsilon}(1+2\epsilon)^{1+\epsilon}}{1+\epsilon}\right)(1+\epsilon)\|(x_r)_1-(x_r)_2\|_C,$$

which by assumptions (11) implies Q_{n1} is a contraction.

We now show that Q_{n2} is compact. To see this, we first prove that Q_{n2} is continuous on B_k. In fact, for $\{(x_r)_m\}\subseteq B_k$ with $(x_r)_m\to x_r$ in B_k, combining (H_5) and (H_6), we see that:

$$g((x_r)_m)\to g(x_r)\text{and}\quad G\left(s,(x_r)_m(h_2(s))\right)\to G\left(s,x_r(h_2(s))\right),\qquad m\to+\infty.$$

Since

$$\left\|G\left(s,(x_r)_m(h_2(s))\right)-G\left(s,x_r(h_2(s))\right)\right\|_{1+2\epsilon}\leq 2g_k(s),$$

we deduce from the dominated convergence theorem and the strong continuity of $W(1+\epsilon)$ that:

$$\|Q_{n2}(x_r)_m - Q_{n2}x_r\|_{1+2\epsilon}$$
$$\leq \sup_{\epsilon\geq -1} \Big(\Big\|W(1+\epsilon)W\Big(\frac{1}{n}\Big)[g((x_r)_m) - g(x_r)]\Big\|_{1+2\epsilon}$$
$$+ \Big\|\int_0^{1+\epsilon} W(1+\epsilon-s)\int_0^s \gamma(s-\tau)\Big[F\Big(\tau,(x_r)_m(h_1(\tau))\Big)$$
$$- F\Big(\tau,x_r(h_1(\tau))\Big)\Big]d\tau ds\Big\|_{1+2\epsilon}$$
$$+ \Big\|\int_0^{1+\epsilon} W(1+\epsilon-s)\Big[G\Big(s,(x_r)_m(h_2(s))\Big) - G\Big(s,x_r(h_2(s))\Big)\Big]ds\Big\|_{1+2\epsilon}$$
$$+ \Big\|\int_0^{1+\epsilon} W(1+\epsilon-s)\big[B\big(u_m(s)-u(s)\big)\big]ds\Big\|_{1+2\epsilon} \to 0 \quad as \quad m\to\infty,$$

i.e., Q_{n2} is continuous on B_k.

We then verify that $\{(Q_{n2}x_r)(\cdot), x_r \in B_k\} \subseteq C([0,1+2\epsilon], X_{1+2\epsilon})$ is a family of equicontinuous function. Indeed, let $\epsilon > 0$ be small enough, then:

$$\|(Q_{n2}x_r)(1+2\epsilon) - (Q_{n2}x_r)(1+\epsilon)\|_{1+2\epsilon}$$
$$\leq \|W(1+2\epsilon) - W(1+\epsilon)\|[\|(x_r)_0\|_{1+2\epsilon} + N(1+\epsilon)k]$$
$$+ \|A^{2\epsilon}\|(M_1(1+3\epsilon)\,(1+\epsilon)(k+1) + M\|u_n\| + \|g_k(\cdot)\|_\infty)\int_0^1 \|AW(1+2\epsilon-s)$$
$$- AW(1+\epsilon-s)\|ds$$
$$+ (M_1(1+3\epsilon)\,(1+\epsilon)(k+1) + M\|u_n\| + \|g_k(\cdot)\|_\infty)\int_1^{1+\epsilon} \Big(\frac{C_{1+2\epsilon}}{(1+2\epsilon-s)^{1+2\epsilon}}$$
$$+ \frac{C_{1+2\epsilon}}{(1+\epsilon-s)^{1+2\epsilon}}\Big)ds$$

$$+(M_1(1+3\epsilon)(1+\epsilon)(k+1) + M\|u_n\| + \|g_k(\cdot)\|_\infty)\int_{1+\epsilon}^{1+2\epsilon} \frac{C_{1+2\epsilon}}{(1+2\epsilon-s)^{1+2\epsilon}}ds$$
$$\leq W(1+2\epsilon) - W(1+\epsilon)[\|(x_r)_0\|_{1+2\epsilon} + N(1+\epsilon)k]$$
$$+ (M_1(1+3\epsilon)(1+\epsilon)(k+1) + M\|u_n\| + \|g_k(\cdot)\|_\infty)$$
$$\times [\|A^{2\epsilon}\|\int_0^1 \|AW(1+2\epsilon-s) - AW(1+\epsilon-s)\|ds + \frac{C_{1+2\epsilon}}{2\epsilon}(2\epsilon)^{-2\epsilon} + \epsilon^{-2\epsilon}.$$

It is easy to see that, as $\epsilon \to 0$ the right-hand side of the above inequality tends to be zero independently of $x_r \in B_k$, since $\big(W(1+\epsilon)\big)_{\epsilon\geq 0}$ and $\big(A^{1+2\epsilon}W(1+\epsilon)\big)_{\epsilon\geq 0}$ are continuous in t in the uniform operator

topology by Lemma 2.1. Analogously, we can deduce that the functions $\{Q_{n2}x_r, x_r \in B_k\}$ are equicontinuous at $\epsilon = -1$. So, Q_{n2} maps B_K into a family of equicontinuous functions.

Now, we show that for fixed $\epsilon \geq -1$, the set $\{(Q_{n2}x_r)(1+\epsilon), x_r \in B_k\}$ is relatively compact in $X_{1+2\epsilon}$.

If $\epsilon = -1$, then $(Q_{n2}x_r)(0) = (x_r)_0 - W\left(\frac{1}{n}\right)g(x_r)$. As $g((x_r))$ is a bounded set on B_k, from the compactness of $W\left(\frac{1}{n}\right)$ and the closedness of $A^{1+2\epsilon}$, we see readily that the assertion is true for $\epsilon = -1$.

For $\epsilon \geq -1$, let us write $(Q_{n2}x_r)(1+\epsilon) = W(1+\epsilon)\left[(x_r)_0 - W\left(\frac{1}{n}\right)g(x_r)\right] + \left(Q^1_{n^2}x_r\right)(1+\epsilon)$ with

$$
\begin{aligned}
&(Q^1_{n2}x_r)(1+\epsilon) \\
&\qquad = \int_0^{1+\epsilon} W(1+\epsilon-s)\left(G\left(s, x_r\left(h_2(s)\right)\right) + Bu_n(s)\right)ds \\
&\qquad - \int_0^{1+\epsilon} W(1+\epsilon-s)\int_0^s \gamma(s-\tau)F\left(\tau, x_r\left(h_1(\tau)\right)\right)d\tau ds.
\end{aligned}
$$

Then, taking $-1 < \epsilon < 0$, we have the following estimate:

$$
\begin{aligned}
&\|A^{1+2\epsilon}(Q^1_{n2}x_r)(1+\epsilon)\| \\
&\qquad \leq \left\| \int_0^{1+\epsilon} A^{1+2\epsilon}W(1+\epsilon-s)\left(G\left(s, x_r\left(h_2(s)\right)\right) + Bu_n(s)\right)ds \right\| \\
&\qquad + \left\| \int_0^{1+\epsilon} A^{1+2\epsilon}W(1+\epsilon-s)\int_0^s \gamma(s-\tau)F\left(\tau, x_r\left(h_1(\tau)\right)\right)d\tau\, ds \right\| \\
&\qquad \leq -\frac{C_{1+2\epsilon}(1+2\epsilon)^{-2\epsilon}}{2\epsilon}\left(\|g_k(\cdot)\|_\infty + M\|u_n\|\right) \\
&\qquad - \frac{C_{1+2\epsilon}(1+2\epsilon)^{1-2\epsilon}M_1(1+\epsilon)}{2\epsilon}\left(\|x_r\|_{1+\epsilon} + 1\right),
\end{aligned}
$$

which means $\{A^{1+2\epsilon}(Q^1_{n2}x_r)(1+\epsilon), x_r \in B_k\}$ is bounded in X. Hence, we conclude immediately that $(Q^1_{n2}x_r)(1+\epsilon)$ is relatively compact in $X_{1+\epsilon}$ by the compactness of operator $A^{-(1+2\epsilon)}: X \to X_{1+2\epsilon}$, since the imbedding $X_{1+2\epsilon} \hookrightarrow X_{1+\epsilon}$ is compact. Thus, $(Q_{n2}x_r)(1+\epsilon)$ is also relatively compact in $X_{1+\epsilon}$ because $W(1+\epsilon)\left[(x_r)_0 - W\left(\frac{1}{n}\right)g(x_r)\right]$does so. Therefore, from the infinite-dimensional version of the Ascoli–Arzela theorem, Q_{n2} is a completely continuous operator on $C([0, 1+2\epsilon], X_{1+\epsilon})$. Those arguments enable us to conclude that $Q_n = Q_{n1} + Q_{n2}$ is a condensing map on B_k. Consequently, according to Theorem 2.2, Q_n has a fixed point $(x_r)_n(\cdot)$ on B_k, which is a mild solution $(x_r)_n(\cdot)$ of equation (12), and this ends the proof of this lemma.

Now, we introduce the solution sets D and $D(1+\epsilon)$, respectively, as:

$$D = \{(x_r)_n \in C([0,1+2\epsilon], X_{1+\epsilon}) : (x_r)_n = Q_n(x_r)_n, n \geq 1\},$$

$$D(1+\epsilon) = \{(x_r)_n(1+\epsilon) : (x_r)_n \in D, n \geq 1\}, \qquad \epsilon \geq -1.$$

For them, we have that (see [1]):

Lemma 3.2 Assume that all the conditions of Theorem 3.1 are fulfilled. Then, for each $\epsilon \geq -1, D(1+\epsilon)$ is relatively compact in $X_{1+\epsilon}$ and D is equicontinuous on $(0, 1+2\epsilon]$.

Proof. For $n \geq 1, (x_r)_n \in D$ and $\epsilon \geq -1$, we can derive that:

$$\begin{aligned}(x_r)_n(1+\epsilon) = W(1+\epsilon)\left[(x_r)_0 + F\left(0,(x_r)_n\left(h_1(0)\right)\right) - W\left(\frac{1}{n}\right)g((x_r)_n)\right] \\ - F\left(1+\epsilon,(x_r)_n\left(h_1(1+\epsilon)\right)\right) \\ + \int_0^{1+\epsilon} W(1+\epsilon-s)\left[AF\left(s,(x_r)_n\left(h_1(s)\right)\right) - \int_0^s \gamma(s-\tau)F\left(\tau,(x_r)_n\left(h_1(\tau)\right)\right)d\tau\right. \\ \left. + G\left(s,(x_r)_n\left(h_2(s)\right)\right) + Bu_n(s)\right]ds = \sum_{i=1}^{3} I_i.\end{aligned}$$

Since it follows from (H_3) and (H_6) that $F\left(1+\epsilon,(x_r)_n\left(h_1(1+\epsilon)\right)\right)$ and $g\left((x_r)_n(1+\epsilon)\right)$ are bounded in $X_{1+\epsilon}$ uniformly in $n \in \mathbb{N}$, it is clear that I_1, I_2 are both relatively compact in $X_{1+\epsilon}$ as $\left(W(1+\epsilon)\right)_{\epsilon\geq 0}$ and $A^{-(1+\epsilon)}$ are compact operators. Now, we verify I_3 is relatively compact in $X_{1+\epsilon}$ too. For this, taking $-1 < \epsilon < 0$ as above, we calculate similarly that:

$$\begin{aligned}(A^{1+\epsilon}I_3) \leq \left\|\int_0^{1+\epsilon} A^{1+\epsilon}W(1+\epsilon-s)A^{1+2\epsilon}F\left(s,(x_r)_n\left(h_1(s)\right)\right)ds\right\| \\ + \left\|\int_0^{1+\epsilon} A^{1+\epsilon}W(1+\epsilon-s)\int_0^s \gamma(s-\tau)\left[F\left(\tau,(x_r)_n\left(h_1(\tau)\right)\right)\right]d\tau ds\right\| \\ + \left\|\int_0^{1+\epsilon} A^{1+\epsilon}W(1+\epsilon-s)\left(G\left(s,(x_r)_n\left(h_2(s)\right)\right) + Bu_n(s)\right)ds\right\| \\ \leq \frac{(1+2\epsilon)^{\epsilon}C_{1+\epsilon}}{\epsilon}(1+\epsilon)(\|(x_r)\|_{1+\epsilon}+1) - \frac{(1+2\epsilon)^{-\epsilon}C_{1+\epsilon}}{\epsilon}M_1(1+\epsilon)(\|(x_r)\|_{1+\epsilon}+1) \\ - \frac{(1+2\epsilon)^{-\epsilon}C_{1+\epsilon}}{\epsilon}(\|g_k(\cdot)\|_{\infty} + M\|u_n\|).\end{aligned}$$

This shows that $\{A^{1+\epsilon}I_3, (x_r)_n \in D(1+\epsilon)\}$ is uniformly bounded in X and I_3 is relatively compact in $X_{1+\epsilon}$. Thus, $D(1+\epsilon)$ is relatively compact in $X_{1+\epsilon}$.

The second assertion can be proved by the similar arguments as verifying the equicontinuity of $\{(Q_{n2}(x_r))(1+\epsilon), x_r \in B_k\}$ in the proofs of Lemma 3.1.

Now, we prove Theorem 3.1 (see [1]).

Proof of Theorem 3.1. We prove that the solution set D is relatively compact in $C([0,1+2\epsilon], X_{1+\epsilon})$. Due to Lemma 3.2, it remains to verify that $D(0)$ is relatively compact in $X_{1+\epsilon}$ and D is equicontinuous at $\epsilon = -1$.

For $(x_r)_n \in D, n \geq 1$, let

$$\overline{(x_r)_n}(1+\epsilon) = \begin{cases} (x_r)_n(1+2\epsilon), & \epsilon \geq -1, \\ (x_r)_n(1+2\epsilon), & \epsilon \geq 0, \end{cases}$$

where δ comes from the condition (H_6). Then, by $(H_6), g((x_r)_n) = g((x_r)_n)$.

By virtue of Lemma 3.2, D is relatively compact in $C\big((0,1+2\epsilon], X_{1+\epsilon}\big)$. Hence, there exists a subsequence of $\{(x_r)_n\}$, denoted by itself, that converges to some function $x_r(\cdot)$ in $C\big((0,1+2\epsilon], X_{1+\epsilon}\big)$, as $n \to \infty$. Thus, by the continuity of g, the strong continuity of $W(1+\epsilon)$ at $\epsilon = -1$ and the fact $g((x_r)_n) = g\big(\overline{(x_r)_n}\big)$, we find:

$$\begin{aligned} &\|(x_r)_n(0) - ((x_r)_0 - g(x_r))\|_{1+\epsilon} \\ &\qquad \leq \left\|W\left(\frac{1}{n}\right) g((x_r)_n) - W\left(\frac{1}{n}\right) g(x_r)\right\| (x_r)_{1+\epsilon} + \left\|W\left(\frac{1}{n}\right) g(x_r) - g(x_r)\right\|_{1+\epsilon} \\ &\qquad = \left\|W\left(\frac{1}{n}\right)[g((x_r)_n) - g(x_r)]\right\|_{1+\epsilon} + \left\|\left[W\left(\frac{1}{n}\right) - I\right] g(x_r)\right\|_{1+\epsilon} \to 0 \; as \; n \to \infty, \end{aligned}$$

i.e., $D(0)$ is relatively compact in $X_{1+\epsilon}$.

On the other hand, for $\epsilon \geq -1$,

$$\|(x_r)_n(1+\epsilon)-(x_r)_n(0)\|_{1+\epsilon}$$
$$= \|W(1+\epsilon)\left[(x_r)_0 + F\left(0,(x_r)_n(h_1(0))\right) - W\left(\frac{1}{n}\right)g((x_r)_n)\right]$$
$$- F\left(1+\epsilon,(x_r)_n(h_1(1+\epsilon))\right)$$
$$+\int_0^{1+\epsilon} W(1+\epsilon-s)\left[AF\left(s,(x_r)_n(h_1(s))\right) - \int_0^s \gamma(s-\tau)F\left(\tau,(x_r)_n(h_1(\tau))\right)d\tau\right.$$
$$\left.+ G\left(s,(x_r)_n(h_2(s))\right) + Bu_n(s)\right]ds$$
$$- \left[(x_r)_0 + F\left(0,(x_r)_n(h_1(0))\right) - W\left(\frac{1}{n}\right)g((x_r)_n) - F\left(0,(x_r)_n(h_1(0))\right)\right\|_{1+\epsilon}$$

$$\leq \left\|\int_0^{1+\epsilon} W(1+\epsilon-s)\left[AF\left(s,(x_r)_n(h_1(s))\right) - \int_0^s \gamma(s-\tau)F\left(\tau,(x_r)_n(h_1(\tau))\right)d\tau\right.\right.$$
$$\left.\left.+ G\left(s,(x_r)_n(h_2(s))\right) + Bu_n(s)\right]ds\right\|_{1+\epsilon} + \|W(1+\epsilon)(x_r)_0-(x_r)_0\|_{1+\epsilon}$$
$$+ \left\|(W(1+\epsilon)-I)W\left(\frac{1}{n}\right)g((x_r)_n)\right\|_{1+\epsilon} + \left\|(W(1+\epsilon)-I)F\left(0,(x_r)_n(h_1(0))\right)\right\|_{1+\epsilon}$$
$$+ \left\|F\left(1+\epsilon,(x_r)_n(h_1(1+\epsilon))\right) - F\left(0,(x_r)_n(h_1(0))\right)\right\|_{1+\epsilon} \to 0,$$

uniformly in n as $\epsilon \to -1$, since $D(0) = \left\{(x_r)_0 - W\left(\frac{1}{n}\right)g((x_r)_n): (x_r)_n \in D\right\}_{n=1}^{\infty}$ is relatively compact and $A^{-(1+\epsilon)}$ is compact. Thus, we infer that the set $D \subseteq C([0,1+2\epsilon],X_{1+\epsilon})$ is equicontinuous at

$\epsilon = -1$, and so D is relatively compact in $C([0,1+2\epsilon],X_{1+\epsilon})$.

Therefore, for $\{(x_r)_n(\cdot)\} \subset D$, we may find an $x_r^*(\cdot)$ in $C([0,1+2\epsilon],X_{1+\epsilon})$ such that (by pass of subsequence if necessary) $(x_r)_n \to x_r^*$ in $C([0,1+2\epsilon],X_{1+\epsilon})$ as $n \to \infty$. Observe that $(x_r)_n, n \in \mathbb{N}^+$, verify the expression of mild solutions for the equation (12), i. e.:

$$(x_r)_n(1+\epsilon) = W(1+\epsilon)\left[(x_r)_0 + F\left(0,(x_r)_n(h_1(0))\right) - W\left(\frac{1}{n}\right)g((x_r)_n)\right]$$
$$- F\left(1+\epsilon,(x_r)_n(h_1(1+\epsilon))\right) + \int_0^{1+\epsilon} W(1+\epsilon-s)[AF\left(s,(x_r)_n(h_1(s))\right)$$
$$- \int_0^s \gamma(s-\tau)F\left(\tau,(x_r)_n(h_1(\tau))\right)d\tau + G\left(s,(x_r)_n(h_2(s))\right) + Bu_n(s)ds,$$

for $\epsilon \geq -1$, taking limit as $n \to \infty$ on both sides, we then obtain that

$$x_r^*(1+\epsilon) = W(1+\epsilon)\left[(x_r)_0 + F\left(0, x_r^*(h_1(0))\right) - g(x_r^*)\right] - F\left(1+\epsilon, x_r^*(h_1(1+\epsilon))\right)$$
$$+ \int_0^{1+\epsilon} W(1+\epsilon-s)\left[AF\left(s, x_r^*(h_1(s))\right) - \int_0^s \gamma(s-\tau)F\left(\tau, x_r^*(h_1(\tau))\right)d\tau\right.$$
$$\left. + G\left(s, x_r^*(h_2(s))\right) + Bu(s)\right]ds,$$

for $\epsilon \geq -1$, which manifests that Eq. 1 has a mild solution $x_r^*(\cdot)$. The existence result is proved.

Now, we prove the approximate controllability of equation (1). That is (see [1]):

Theorem 3.2 Assume that the assumptions (H_0)–(H_6) are satisfied. Additionally, we suppose that the function $F:[0,1+2\epsilon]\times X_{1+2\epsilon} \to X_{2+3\epsilon}$ uniformly bounded in $X_{2+3\epsilon}$, and $G:[0,1+2\epsilon]\times X_{1+2\epsilon} \to X$ and $g:C([0,1+2\epsilon],X_{1+2\epsilon}) \to X_{1+2\epsilon}$ are both uniformly bounded in X. Then, the equation (1) is approximately controllable on $[0,1+2\epsilon]$.

Proof. Let $\bar{x}_r^{1+2\epsilon} \in X, \lambda \in (0,1)$ and $\bar{x}_r^{\lambda}(\cdot):[0,1+2\epsilon] \to X_{1+2\epsilon}$ is the mild solution of equation (1) obtained in Theorem 3.1 under the control $u(1+\epsilon)$ given by Eq. 10. It means that $\bar{x}_r^{\lambda}(\cdot)$ satisfies:

$$\bar{x}_r^{\lambda}(1+2\epsilon) = W(1+2\epsilon)\left[(x_r)_0 + F\left(0, \bar{x}_r^{\lambda}(h_1(0))\right) - g(\bar{x}_r^{\lambda})\right] - F\left(1+2\epsilon, \bar{x}_r^{\lambda}(h_1(1+2\epsilon))\right)$$
$$+ \int_0^{1+2\epsilon} W(1+2\epsilon-s)\left[AF\left(s, \bar{x}_r^{\lambda}(h_1(s))\right) - \int_0^s \gamma(s-v)F\left(v, \bar{x}_r^{\lambda}(h_1(v))\right)dv\right.$$
$$\left. + G\left(s, \bar{x}_r^{\lambda}(h_2(s))\right) + Bu(s)\right]ds$$

$$= W(1+2\epsilon)\left[(x_r)_0 + F\left(0, \bar{x}_r^{\lambda}(h_1(0))\right) - g(\bar{x}_r^{\lambda})\right] - F\left(1+2\epsilon, \bar{x}_r^{\lambda}(h_1(1+2\epsilon))\right)$$
$$+ \int_0^{1+2\epsilon} AW(1+2\epsilon-s)F\left(s, \bar{x}_r^{\lambda}(h_1(s))\right)ds$$
$$- \int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^s \gamma(s-v)F\left(v, \bar{x}_r^{\lambda}(h_1(v))\right)dvds$$

$$+\int_0^{1+2\epsilon} W(1+2\epsilon-s)\Big\{G\left(s,\bar{x}_r^{\lambda}(h_2(s))\right)$$
$$+BB^*W^*(1+2\epsilon-s)R(\lambda,\Gamma_{1+2\epsilon})\Big[\bar{x}_r^{\lambda}$$
$$-W(1+2\epsilon)\left[(x_r)_0+F\left(0,\bar{x}_r^{\lambda}(h_1(0))\right)-g(\bar{x}_r^{\lambda})\right]+F\left(1+2\epsilon\,,\bar{x}_r^{\lambda}(h_1(1+2\epsilon))\right)$$
$$-\int_0^{1+2\epsilon} AW(1+2\epsilon-v)F\left(v,\bar{x}_r^{\lambda}(h_1(v))\right)dv$$
$$+\int_0^{1+2\epsilon} W(1+2\epsilon-v)\int_0^{v}\gamma(v-\tau)F\left(v,\bar{x}_r^{\lambda}(h_1(v))\right)d\tau dv$$
$$-\int_0^{1+2\epsilon} W(1+2\epsilon-v)G\left(v,\bar{x}_r^{\lambda}(h_2(v))\right)dv\Big]\Big\}ds.$$

From the definition of $\Gamma_{1+2\epsilon}$, it follows that:

$$\bar{x}_r^{\lambda}(1+2\epsilon)=\bar{x}_r^{1+2\epsilon}$$
$$+(\Gamma_{1+2\epsilon}R(\lambda,\Gamma_{1+2\epsilon})-I)\Big[\bar{x}_r^{1+2\epsilon}-W(1+2\epsilon)\left[(x_r)_0+F\left(0,\bar{x}_r^{\lambda}(h_1(0))\right)-g(\bar{x}_r^{\lambda})\right]$$
$$+F\left(1+2\epsilon\,,\bar{x}_r^{\lambda}(h_1(1+2\epsilon))\right)-\int_0^{1+2\epsilon} AW(1+2\epsilon-s)F\left(s,\bar{x}_r^{\lambda}(h_1(s))\right)ds$$
$$+\int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^{s}\gamma(s-v)F\left(v,\bar{x}_r^{\lambda}(h_1(v))\right)dvds$$
$$-\int_0^{1+2\epsilon} W(1+2\epsilon-s)G\left(s,\bar{x}_r^{\lambda}(h_2(s))\right)ds\Big].$$

$$=\bar{x}_r^{1+2\epsilon}+\lambda R(\lambda,\Gamma_{1+2\epsilon})\Big[\bar{x}_r^{1+2\epsilon}-W(1+2\epsilon)\left[(x_r)_0+F\left(0,\bar{x}_r^{\lambda}(h_1(0))\right)-g(\bar{x}_r^{\lambda})\right]$$
$$+F\left(1+2\epsilon,\bar{x}_r^{\lambda}(h_1(1+2\epsilon))\right)-\int_0^{1+2\epsilon} AW(1+2\epsilon-s)F\left(s,\bar{x}_r^{\lambda}(h_1(s))\right)ds$$
$$+\int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^{s}\gamma(s-v)F\left(v,\bar{x}_r^{\lambda}(h_1(v))\right)dvds$$
$$-\int_0^{1+2\epsilon} W(1+2\epsilon-s)G\left(s,\bar{x}_r^{\lambda}(h_2(s))\right)ds\Big].$$

So, we have

$$\left\|\bar{x}_r^{\lambda}(1+2\epsilon)-\bar{x}_r^{1+2\epsilon}\right\|$$

$$=\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\left[\bar{x}_r^{1+2\epsilon}-W(1+2\epsilon)\left[(x_r)_0+F\left(0,\bar{x}_r^{\lambda}\left(h_1(0)\right)\right)-g\left(\bar{x}_r^{\lambda}\right)\right]\right.\right.$$

$$+F\left(1+2\epsilon,\bar{x}_r^{\lambda}\left(h_1(1+2\epsilon)\right)\right)-\int_0^{1+2\epsilon} AW\,(1+2\epsilon-s)F\,\left(s,\bar{x}_r^{\lambda}\left(h_1(s)\right)\right)ds$$

$$+\int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^{s}\gamma(s-v)F\left(v,\bar{x}_r^{\lambda}\left(h_1(v)\right)\right)dvds$$

$$-\int_0^{1+2\epsilon} W(1+2\epsilon$$

$$\left.\left.-s)G\left(s,\bar{x}_r^{\lambda}\left(h_2(s)\right)\right)ds\right]\right\|. \qquad (14)$$

Due to the compactness of $W(1+\epsilon),\epsilon\geq 0$, and the uniform boundedness of g,F, we see that there are subsequences of $W(1+2\epsilon)\left[F\left(0,\bar{x}_r^{\lambda}\left(h_1(0)\right)\right)-g\left(\bar{x}_r^{\lambda}\right)\right]$, still denoted by itself, converging to some $\tilde{x}_r^1\in X$, as $\lambda\to 0^+$.

Since $F\left(1+2\epsilon,\bar{x}_r^{\lambda}\left(h_1(1+2\epsilon)\right)\right)$ is bounded in $X_{2+3\epsilon}$, it is also compact in X from the compactness of $A^{-(2+3\epsilon)}$, and hence there is $(\tilde{x}_r)_2\in X$ such that (by passing to a subsequence if necessary)

$F\left(1+2\epsilon,\bar{x}_r^{\lambda}\left(h_1(1+2\epsilon)\right)\right)\to(\tilde{x}_r)_2\in X$ in $\|\cdot\|$ as $\lambda\to 0^+$.

On the other hand, using the uniform boundedness of $F:[0,1+2\epsilon]\times X_{1+2\epsilon}\to X_{2+3\epsilon}$ and $G:[0,1+2\epsilon]\times X_{1+2\epsilon}\to X$ again, we find that $\left\{F\left(s,\bar{x}_r^{\lambda}\left(h_1(s)\right)\right)\right\}$ and $\left\{G\left(s,\bar{x}_r^{\lambda}\left(h_2(s)\right)\right)\right\}$ are both uniform bounded in $L^2([0,1+2\epsilon],X)$. Thus, there exist subsequences still denoted by $\left\{F\left(s,\bar{x}_r^{\lambda}\left(h_1(s)\right)\right)\right\}$ and $\left\{G\left(s,\bar{x}_r^{\lambda}\left(h_2(s)\right)\right)\right\}$ that converge weakly to say $\tilde{F}(s)$ and $\tilde{G}(s)$ in $L^2([0,1+2\epsilon],X)$, respectively. Furthermore, similarly to the proof of the compactness of Q_{n2} in Lemma 3.1, one can easily verify that the maps $x_r(1+\epsilon)\to\int_0^{1+\epsilon} AW(1+\epsilon-s)x_r(s)ds{:}L^2([0,1+2\epsilon],X_{2+3\epsilon})\to C([0,1+2\epsilon],X)$ and $x_r(1+\epsilon)\to\int_0^{1+\epsilon} W(1+\epsilon-s)x_r(s)ds{:}L^2([0,1+2\epsilon],X)\to C([0,1+2\epsilon],X)$ are compact. As a result, we infer immediately that:

$$\int_0^{1+2\epsilon} AW(1+2\epsilon-s)\left(F\left(s,\bar{x}_r^{\lambda}\left(h_1(s)\right)\right)-\tilde{F}(s)\right)ds\to 0,\qquad as\quad \lambda\to 0^+,$$

and

$$\int_0^{1+2\epsilon} W(1+2\epsilon-s)\left(G\left(s,\bar{x}_r^{\lambda}(h_2(s))\right)-\tilde{G}(s)\right)ds\to 0, \qquad as \quad \lambda\to 0^+.$$

Combining (14) with above assertions, we find that:

$$\begin{aligned}
&\left\|\bar{x}_r^{\lambda}(1+2\epsilon)-\bar{x}_r^{1+2\epsilon}\right\| \\
&\quad\le \left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})(\bar{x}_r^{1+2\epsilon}-W(1+2\epsilon)(x_r)_0-(\tilde{x}_r)_1+(\tilde{x}_r)_2)\right\| \\
&\quad+\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\left(W(1+2\epsilon)\left(F\left(0,\bar{x}_r^{\lambda}(h_1(0))\right)-g(\bar{x}_r^{\lambda})\right)-(\tilde{x}_r)_1\right)\right\| \\
&\quad+\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\left(F\ \left(1+2\epsilon,\bar{x}_r^{\lambda}(h_1(1+2\epsilon))\right)-(\tilde{x}_r)_2\right)\right\| \\
&\quad+\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\int_0^{1+2\epsilon} AW(1+2\epsilon-s)\left(F\left(s,\bar{x}_r^{\lambda}(h_1(s))\right)-\tilde{F}(s)\right)ds\right\| \\
&+\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\int_0^{1+2\epsilon} AW(1+2\epsilon-s)\tilde{F}(s)ds\right\| \\
&\quad+\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^{s}\gamma(s-v)\left(F\left(v,\bar{x}_r^{\lambda}(h_1(v))\right)\right.\right. \\
&\quad\left.\left.-\tilde{F}(v)\right)dvds\right\|+\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\int_0^{1+2\epsilon} W(1+2\epsilon-s)\int_0^{s}\gamma(s-v)\tilde{F}(v)dvds\right\| \\
&+\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\int_0^{1+2\epsilon} W(1+2\epsilon-s)\left(G\left(s,\bar{x}_r^{\lambda}(h_2(s))\right)-\tilde{G}(s)\right)ds\right\| \\
&\quad+\left\|\lambda R(\lambda,\Gamma_{1+2\epsilon})\int_0^{1+2\epsilon} W(1+2\epsilon-s)\tilde{G}(s)ds\right\|\to 0, \qquad as \quad \lambda\to 0^+.
\end{aligned}$$

So $\bar{x}_r^{\lambda}(1+2\epsilon)\to\bar{x}_r^{1+2\epsilon}$ holds in X, which manifests the approximate controllability of system (1). The proof is complete.

4 An Example

As an application of the Theorem 3.2, we consider the approximate controllability for the following neutral integro-differential equation with nonlocal condition (see [1]):

$$\begin{cases} \dfrac{\partial}{\partial(1+\epsilon)}\left[z(1+\epsilon,x_r)+\displaystyle\int_0^{\pi} a(y_r,x_r)\sin\left(z(1+\epsilon,y_r)+\frac{\partial}{\partial y_r}z(1+\epsilon,y_r)\right)dy_r\right]=\frac{\partial^2}{\partial x_r^2}z(1+ \\ +\displaystyle\int_0^{1+\epsilon} b(1+\epsilon-s)\frac{\partial^2}{\partial x_r^2}z(s,x_r)ds+c\left(1+\epsilon,z(\sin 1+\epsilon\,,x_r),\frac{\partial}{\partial x_r}z(\sin 1+\epsilon\,,x_r)\right)+u(1+\epsilon,x_r),0\le \\ z(1+\epsilon,0)=z(1+\epsilon,\pi)=0, \qquad \epsilon\ge -1, \\ z(0,x_r)+\displaystyle\sum_{i=1}^{p}\cos\big(z(1+i\epsilon,x_r)\big)=z_0(x_r), \qquad 0\le x_r\le\pi, \end{cases}$$

where $1+2\epsilon\le\pi,\epsilon>0,p\in\mathbb{N}^+,z_0(x_r)\in X:=L^2([0,\pi]).b(\cdot)$ is continuous, then there exists a constant $M_1>0$ such that $\|b(\cdot)\|\le M_1$.

We assume that the functions $a(\cdot,\cdot),k(\cdot,\cdot)$, and $c(\cdot,\cdot,\cdot)$ satisfy, respectively, the following conditions.

(a_1) The function $a\colon[0,\pi]\times[0,\pi]\to\mathbb{R}$ verifies that:

(i) The function $\frac{\partial^2}{\partial x_r^2}a(y_r,x_r)$ is measurable and

$$N_1=\left[\int_0^{\pi}\int_0^{\pi}\left(\frac{\partial^2}{\partial x_r^2}a(y_r,x_r)\right)^2dy_rdx_r\right]^{\frac{1}{2}}<\infty;$$

(ii) $a(y_r,0)=a(y_r,\pi)=0$ for each $y_r\in[0,\pi]$.

$(a_2)b(\cdot)\in L^1(\mathbb{R}^+)\cap C^1(\mathbb{R}^+)$ with primitive $B(1+\epsilon)\in L^1_{loc}(\mathbb{R}^+)$. Here, $B(1+\epsilon)$ is nonpositive, nondecreasing, and $B(0)=-1$. Moveover, the scalar function $b(\cdot)$ satisfies that $\lambda\in\Lambda$ implies

$\psi(\lambda):=1+b^*(\lambda)\neq 0$, and $\lambda\psi(\lambda)\in\Lambda$, further, $b^*(\lambda)\to 0$ as $|\lambda|\to\infty,\lambda\in\Lambda$.

(a_3) The function $c\colon[0,1+2\epsilon]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous, and there is a constant $C>0$ such that $|c(1+\epsilon,x_r,y_r)|\le C$, for any $(1+\epsilon,x_r,y_r)\in[0,1+2\epsilon]\times\mathbb{R}\times\mathbb{R}$.

Now, we need to rewrite equation (15) in the form of Eq. 1 and verify all the conditions of Theorem 3.2. To this, let $\big(A,D(A)\big)$ be defined by:

$$Az=-z''$$

with the domain:

$$D(A)=\{z(\cdot)\in X\colon z',z''\in X, \qquad \text{and } z(0)=z(\pi)=0\}.$$

It is well known that $-A$ generates a strongly continuous semigroup $\big(S(1+\epsilon)\big)_{\epsilon\ge -1}$ which is compact, analytic, and self-adjoint. Furthermore, A has a discrete spectrum, the eigenvalues are $n^2,n\in\mathbb{N}$, with

the corresponding normalized eigenvectors $\xi_n(x_r) = \sqrt{\frac{2}{\pi}}\sin(nx_r)\,, n = 1, 2, \ldots$. Then, the following properties hold:

(i) If $z \in D(A)$, then

$$Az = \sum_{n=1}^{\infty} n^2\langle z, \xi_n\rangle\xi_n.$$

(ii) For each $z \in X$,

$$A^{-\frac{1}{2}}z = \sum_{n=1}^{\infty} \frac{1}{n}\langle z, \xi_n\rangle\xi_n.$$

In particular, $\left\|A^{-\frac{1}{2}}\right\| = 1$.

(iii) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}z = \sum_{n=1}^{\infty} n\langle z, \xi_n\rangle\xi_n$$

on the space

$$D\left(A^{\frac{1}{2}}\right) = \left\{z(\cdot) \in X, \sum_{n=1}^{\infty} n\langle z, \xi_n\rangle\xi_n \in X\right\}.$$

We know that, for the operator $\left(-A, D(A)\right)$, there is $\varphi \in (0, \pi/2)$ such that:

$$\Lambda := \left\{\lambda \in C: |\arg\lambda| < \frac{\pi}{2} + \varphi\right\} \subset \rho(-A),$$

where $\rho(-A)$ denotes the resolvent set of $-A$. Then, from (a_2) the conditions (V_1')–(V_3') are verified, and hence the linear equation for equation (15) has a family of resolvent operators $\left(W(1+\epsilon)\right)_{\epsilon\geq -1}$, which is given by, for $x_r \in X$:

$$W(1+\epsilon)x_r = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda(1+\epsilon)}\rho(\lambda)x_r d\lambda = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda(1+\epsilon)}\psi^{-1}(\lambda)(\lambda\psi^{-1}(\lambda) + A)^{-1}x_r d\lambda,$$

where Γis described in Section 2. It is readily seen that $W(1+\epsilon)$ is compact for all $\epsilon \geq 0$, since $R(\lambda, -A)$ is compact for any $\lambda \in \rho(A)$. So, the assumption (H_1) holds for all $\epsilon \geq 0$.

Here, we choose $\epsilon = 0, \gamma(1+\epsilon) = b(1+\epsilon)A$, and define respectively the functions $F: [0, 1+2\epsilon] \times X_{1\,2} \to D(A), G: [0, 1+2\epsilon] \times X_{\frac{1}{2}} \to X$ and $g: C\left([0, 1+2\epsilon], X_{\frac{1}{2}}\right) \to X_{\frac{1}{2}}$ as:

$$F(1+\epsilon,z)(x_r) = F(z)(x_r) = \int_0^{\pi} a(y_r,x_r)\sin\big(z(y_r)+z'(y_r)\big)\,dy_r,$$

$$G(1+\epsilon,z)(x_r) = c\big(1+\epsilon,z(x_r),z'(x_r)\big)$$

and

$$g\big(z(1+\epsilon,x_r)\big) = g(z)(x_r) = \sum_{i=1}^{p} \cos\big(z(1+i\epsilon,x_r)\big).$$

Additionally, we put $\gamma(\cdot) = b(\cdot)A$ and $B = I$. Then, equation (15) can be well rewritten as the abstract form (1).

Let us examine now that for this system the conditions in Theorem 3.2 are all fulfilled. First, it is clear that the operator B satisfies (H_2) and the family $\gamma(\cdot)$ satisfies condition (H_4). Moreover, from the relation (cf. [42]),

$$\|z_2 - z_1\| \le \|z_2 - z_1\|_{\frac{1}{2}},$$

one can easily see that $F: X_{\frac{1}{2}} \to X_1$ is Lipschitz continuous and satisfies the assumption (H_3). Furthermore, for any $(\cdot) \in X_{\frac{1}{2}}$,

$$\|F(z)\|_1^2 = \|AF(z)\|^2 = \sum_{n=1}^{\infty} \left|\left\langle \int_0^{\pi} \frac{\partial^2 a(y_r,x_r)}{\partial x_r^2}\sin\big(z(y_r)+z'(y_r)\big)\,dy_r, \xi_n\right\rangle\right|^2$$

$$\le \pi \int_0^{\pi}\int_0^{\pi} \left(\frac{\partial^2 a(y_r,x_r)}{\partial x_r^2}\right)^2 dy_r dx_r,$$

it then follows from (a_1) that F is uniformly bounded. The assumption (a_3) guarantees that G satisfies (H_5) and is uniformly bounded. The function g is also uniformly bounded and verifies the condition (H_6). Finally, according to ([24] Lemma 4.1), the linear system corresponding to Eq. 15 is approximately controllable on [0, T]; hence, (H_0) holds too. Thus, all the conditions of Theorem 3.2 are well satisfied. Consequently, from Theorem 3.2, equation (15) is approximately controllable on $[0, 1+2\epsilon]$.

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