Shendi University Journal of Applied Science, Issue (10) , June 2023 (4): p 43-57

Sobolev Embeddings and Capacity on Metreic Spaces

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Abstract:

The main aim of this study is to know the Sobolev embeddings into spaces of Campanato and Hölder type with scaling invariant Sobolev-Lorentz capacity on metric spaces and in the Euclidean setting. We develop a capacity theory based on the definition of Sobolev functions on metric spaces with a Borel regular outer measure, with respect to the Lorentz norm .The basic properties of capacity including monotonicity countable (finite) are studied. The researcher used the historical mathematical and analytical method to access related informations to Sobolev embeddings and capacity on metreic spaces**.** The most important result of the study they are embedded inside the paper and refer to theorems of Sobolev embeddings and capacity on metreic and $\mathbb{R}^n$ spaces .

المستخلص :

الهدف الأساسي لهذه الدراسة هو التعرف علي طمر سوبوليف في فضاءات كامباناتو ونوع هولدر مع سعة سوبوليف- لورنتز اللامتغيرة الترقيم علي الفضاءات المترية وفي الوضع الإقليدى .وتم تطوير نظرية السعة بناءاً علي تعريف دوال سوبوليف في الفضاءات المترية بواسطة مقياس بوريل الخارجي المنتظم مع مراعاة معيار لورنتز . وأيضا تمت دراسة الخصائص الأساسية للسعة بما في ذلك الرتابة القابلة للعد (المنتهية) . إستخدم الباحث المنهج الرياضي و التحليلي التاريخي للوصول فيما يتعلق بمعلومات طمر سوبوليف والسعة في الفضاءات المترية . ومن أهم النتائج التي توصلت اليها الدراسة هي متضمنة داخل الورقة وتشير الي نظريات سوبوليف والسعة في الفضاءات المترية والنونية .

Keywords: Campanato spaces , Morrey spaces , Hölder spaces ,Sobolev embeddings, Capacity, Metric spaces.

1.Introduction:

Necessary and sufficient conditions on a rearrangement-invariant Banach function space $X(Q)$ on a cube Q in $\mathbb{R}^n, n \geq 2$, are given for the corresponding Sobolev space $W^1X(Q)$ to be continuously embedded into (generalized) Campanato, Morrey, or Hölder spaces. As an application **,**we prove that each Sobolev function has a quasicontinuous representative. For doubling measures **,**we provide sharp estimates for the capacity of balls. Capacity and Hausdorff measures are related under an additional regularity assumption on the measure, are studied. We develop a capacity theory based on the definition of Sobolev

functions on $\square^n$ with respect to the Lorentz norm. Basic properties of capacity, including monotonicity, finite subadditivity .

2. Campanato with Morrey, and Hölder Type.

We establish Sobolev embedding, in the spirit of Morrey's theorem, into the spaces $C^{0,\varphi}(Q)$, $L^C_\varphi(Q)$, and $L^M_\alpha(Q)$ In fact, the aim is to determine optimal summability conditions on the gradient of a weakly differentiable function on *Q* for such a function to belong to each of these space.

Lemma (2.1). Let $X(Q)$ be an r.i. space on the cube $Q \subset \mathbb{R}^n$.

(i) If $u \in W_0^1 X(Q)$, then u^* is locally absolutely continuous and

$$nC_n^{1/n} \| s^{1/n'} \left(-\frac{du^*}{ds} \right) \|_{\tilde{X}(0,1)} \leq \| \nabla u \|_{X(Q)} \tag{1}$$

Here $C_n = \pi^{n/2} / \Gamma(1+n/2)$, the measure of the unit ball in Rn. Equality holds in (3) if u is radially decreasing.

(ii) If $u \in W^1 X(Q)$, then u° is locally absolutely continuous and there exists a constant K_n depending only on n , such that

$$K_n \| \min\{s, 1-s\}^{1/n'} \left(-\frac{du^\circ}{ds} \right) \|_{\tilde{X}(0,1)} \leq \| \nabla u \|_{X(Q)} \tag{2}$$

Let f be a real-valued measurable function on *Q*. The non increasing rearrangement f^* of f is given by

$$f^*(t) = \sup\{ s > 0 : |\{ x \in Q : |f(x)| > s \}| > t \}, 0 < t < 1$$

and the signed non increasing rearrangement f° of f is given by

$$f^\circ(t) = \sup\{ s \in \square : |\{ x \in Q : f(x) > s \}| > t \}, 0 < t < 1 \tag{3}$$

A Banach space $X(Q)$ of functions defined on *Q*, equipped with the norm $\|\cdot\|_{X(Q)}$ is said to be rearrangement-invariant if the following five axioms hold:

(i) $0 \leq g \leq f$ a.e implies $\| g \|_{X(Q)} \leq \| f \|_{X(Q)}$.

(ii) $0 \leq f_n \leq f$ a.e implies $\| f_n \|_{X(Q)} \leq \| f \|_{X(Q)}$.

(iii) $\|1\|_{X(Q)} < \infty$.

(iv) A constant *C* exists such that $\int_Q f \leq \| f \|_{X(Q)}$ for every $f \in X(Q)$

(v) $\| g \|_{X(Q)} = \| f \|_{X(Q)}$ whenever $f^* = g^*$.

Given an r.i. space $X(Q)$,the set

$$X'(Q) = \left\{ f \text{ measurable on } Q \int_Q |fg| < \infty \text{ for every } g \in X(Q) \right\}$$

equipped with the norm

$$\| f \|_{X'(Q)} = \sup_{\|g\|_{X(Q)} \leq 1} \int_Q |fg| \tag{4}$$

is called the associate space of $X(Q)$. It turns out that $X(Q)$ is again an r.i. space and $X''(Q) = X(Q)$. Furthermore, the Hölder type inequality

$$\int_Q |fg| \le \|f\|_{X(Q)} \|g\|_{X'(Q)}$$

holds for every f and g. For every r.i. space $X(Q)$, there exists a unique r.i. space $\bar{X}(0,1)$ on $(0,1)$, satisfying $\|f\|_{X(Q)} = \|f^*\|_{\bar{X}(Q)}$. such a space, endowed with the norm

$$\|f\|_{\bar{X}(Q)} = \sup_{\|g\|_{X(Q)} \le 1} \int_0^1 f^*(t) g^*(t) dt$$

is called the representation space of $X(Q)$. A property which will be used in our proofs is that the dilation operator E_t, defined for $0 < t \le 1$ by $E_t \varphi(s) = \varphi(st), s \in (0,1)$ at a measurable function φ on (0,1), is bounded on every r.i. space on (0,1) . For a comprehensive treatment of r.i. spaces we refer the reader to [3]. The first-order Sobolev space $W^1X(Q)$ built upon an r.i. space $X(Q)$ is defined as $W^1X(Q) = \{u : u$ is a real-valued weakly differentiable function on $Q\}$ and

$$|\nabla u| \in X(Q)\}.$$

Lemma (2.2). Let G be a measurable subset of $\mathbb{R}^n, n \ge 1$, having finite measure, and let $f \in L^1(G)$. Then

$$\min_{c \in \mathbb{R}} \int_G |f(x) - c| dx = \int_G |f(x) - f^\circ\left(\frac{|G|}{2}\right)| dx \qquad (5)$$

where f° is defined as in (3), with Q replaced by G .

Proof. We prove (5) under the assumption that f is nonnegative. The general case can be dealt with analogously, on splitting f into its positive and negative parts. The fact that the minimum in (5) is attained is an easy consequence of the dominated convergence theorem for the Lebesgue integral.

Now, since

$$f(x) = \int_0^\infty \chi_{\{f > t\}}(x) dt \quad \text{for} \quad x \in G$$

then, for every $c \in \mathbb{R}$,

$$\begin{aligned}
\int_G |f(x) - c| dx &= \int_{\{f > c\}} \left(\int_0^\infty \chi_{\{f > t\}}(x) dt - c \right) dx \\
&+ \int_{\{f \le c\}} \left(c - \int_0^\infty \chi_{\{f > t\}}(x) dt \right) dx \\
&= \int_0^\infty \int_G \chi_{\{f > t\}}(x) \, dx \, dt - c\,|f > c| \\
&+ c(|G| - |\{f > c\}|) - \int_0^\infty \int_G \chi_{\{f \le c\}}(x) \, dx \, dt \\
&= \int_c^\infty |\{f > t\}| dt - \int_0^c |\{f > t\}| dt + c\,|G|
\end{aligned}$$

Thus, $\int_G |f(x)-c|dx$ is a locally absolutely continuous function of c in $\mathbb{R}$ whose derivative equals to $|G|-2|\{f > c\}|$ for a.e. $c \in \mathbb{R}$. Hence, (5) follows , in general, $f^*(|G|/2)$ need not be the unique minimizer in (5).

Lemma (2.3). Let $X(Q)$ be an r.i. space. Then

$$\frac{1}{s}\| r^{1/n}\chi_{(0,s)}(r)\|_{\tilde{X}(0,1)} \le s^{-1/n'}\|\chi_{(0,1)}, s \in \left(0,\frac{1}{2}\right) \tag{6}$$

Proof. Given $s \in (0,1/2)$, we have

$$\begin{aligned}\frac{1}{s}\| r^{1/n}\chi_{(0,s)}(r)\|_{\tilde{X}(0,1)} &\le s^{-1/n'}\|\chi_{(0,1)}(r)\|_{\tilde{X}(0,1)}, s \in \left(0,\frac{1}{2}\right)\\ &\le 2^{-1/n'}\|(s+r)^{-1/n'}\chi_{(0,s)}(r)\|_{\tilde{X}(0,1)}\\ &\le 2^{-1/n'}\|(s+r)^{-1/n'}\chi_{(0,1-s)}(r)\|_{\tilde{X}(0,1)}\\ &\le 2^{-1/n'}\|r^{-1/n'}\chi_{(s,1)}(r)\|_{\tilde{X}(0,1)}\end{aligned}$$

Example(2.4). Assume that α ∈ (-∞, 1], α≠0, and $\varphi(t)=t^{\alpha}$. Then $\theta(t)\approx t^{(1-\alpha)/n}$ when $M_{\theta}(Q)=L_{n/(1-\alpha),\infty}(Q)$ a classical Marcinkiewicz space.

$$W^1 L_{n/(1-\alpha),\infty}(Q) \to L^C_{\varphi}(Q)$$

and $L_{n/(1-\alpha),\infty}(Q)$ is optimal, that is, the largest such r.i. space. Moreover, if $\alpha \in (0,1)$, then

$$W^1 Z_{t^{\alpha}}(Q) \to C^{0,\varphi}(Q)$$

where $Z_{t^{\alpha}}(Q)$ is optimal. Since $\psi \approx \varphi$.

If $\alpha \in (-\infty,0)$, then

$$W^1 Y_{t^{\alpha}}(Q) \to L^M_{\varphi}(Q)$$

with $Y_{t^{\alpha}}(Q)$ optimal, and, since $\omega \approx \varphi$ and $L^C_{\varphi}(Q)=L^M_{\varphi}(Q)$, we get

$$Y_{t^{\alpha}}(Q)=L_{n/(1-\alpha),\infty}(Q).$$

3. Sobolev Capacity on Metric Spaces.

We develop a capacity theory based on the definition of Sobolev functions on metric spaces due to Hajtasz [290]. His definition makes use of the fact that a smooth real-valued function *u* on $\mathbb{R}^n$satisfies

$$|u(x)-u(y)| \le C|x-y|(\mu|\nabla u|(x)+\mu|\nabla u|(y)) \tag{7}$$

for every $x, y \in \mathbb{R}^n$, where μf is the Hardy-Littlewood maximal operator of a locally integrable function *f* defined by

$$\mu f(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|dy$$

Sobolev space (3.1).Let (X, d) be a metric space and let μ be a non-negative Borel regular out measure on X . In the following, we keep the triple (X, d, μ) fixed, and for short, we denote it by X . Let $1<p<\infty$. The $L^p(X)$ is the Banach space of all μ-a.e. defined μ-measurable functions $u: X \to [-\infty,\infty]$ for which the norm

$$\| u \|_{L^p(X)} = \left(\int_X |u|^p \, d\mu \right)^{1/p}$$

is finite. Suppose that $u : X \to [-\infty, \infty]$ is μ-measurable. We denote by $D(u)$ the set of all μ-measurable functions $g : X \to [0, \infty]$ such that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \quad (8)$$

for every $x, y \in X \setminus F, x \neq y$, with $\mu(F) = 0$. Note that the right hand side of (8) is always defined for $x \neq y$. At the points $x, y \in X$, $x \neq y$, where the left hand side of (8) is undefined we may assume that the right hand side is $+\infty$. Following the original definition due to Hajtasz, the Dirichlet space $L^{1,p}(X)$ consists of all μ-measurable functions u with $D(u) \cap L^p(X) \neq \varphi$; the space $L^{1,p}(X)$ is endowed with the semi norm

$$\| u \|_{L^{1,p}(X)} = \inf \left\{ \| g \|_{L^p(X)} : g \in D(u) \cap L^p(X) \right\} \quad (9)$$

The Sobolev space $W^{1,p}(X)$ is the space of all functions $u \in L^p(X)$ for which $D(u) \cap L^p(X) \neq \varphi$. It is clear that (9) defines a seminorm in $W^{1,p}(X)$ .An application of the uniform convexity of $L^p(X)$ implies that there is a unique minimizer of (9); this means that the infimum is attained by a unique function in $D(u) \cap L^p(X)$ .We equip the Sobolev space $W^{1,p}(X)$ with the norm

$$\| u \|_{W^{1,p}(X)} = \left(\| u \|^p_{L^p(X)} + \| u \|^p_{L^{1,p}(X)} \right)^{1/p} \quad (10)$$

Then $W^{1,p}(X)$ is a linear space.

Theorem (3.2). The Sobolev *p*-capacity is an outer measure.

Proof. Clearly $C_p(\varphi) = 0$ and the definition of the capacity implies monoonicity.To prove countable subadditivity, suppose that $E_i, i = 1, 2, 3, \ldots$. are subsets of X . Let $\varepsilon > 0$. We may assume that $\sum_{i=1}^{\infty} C_p(E_i) < \infty$**.** Next we choose $u_i \in A(E_i)$ and $g_{u_i} \in D(u_i) \cap L^p(X)$ so that $\| u_i \|^p_{L^p(X)} + \| g_{u_i} \|^p_{L^p(X)} \leq C_p(E_i) + \varepsilon 2^{-i}$ for $i = 1, 2, 3, \ldots$**.** We show that $v = \sup_i u_i$ is admissible for $\bigcup_{i=1}^{\infty} E_i$ and . First we observe that $v, g \in L^p(X)$ **.**Then we define $v_k = \max_{1 \leq i \leq k} u_i$. the function $g_{v_k} = \max_{1 \leq i \leq k} g_{u_i}$ belongs to $D(v_k) \cap L^p(X)$ **.**Since $g = \sup_i g_{u_i} \in D(v) \cap L^p(X)$ $v_k \to v$ **,** μ-a.e. and $g_{v_k} \to g$ **,** μ-a.e. Clearly $v \geq 1$ in a neighborhood of $\bigcup_{i=1}^{\infty} E_i$. This implies that

$$C_p\left(\bigcup_{i=1}^{\infty} E_i \right) \leq \| v \|^p_{W^{1,p}(X)} \sum_{i=1}^{\infty} (\| u_i \|^p_{L^p(X)} + \| g_{u_i} \|^p_{L^p(X)}) \leq \sum_{i=1}^{\infty} C_p(E_i) + \varepsilon$$

The claim follows by letting $\varepsilon \to 0$.

Theorem (3.3). If $Q_1 \subset Q_2 \subset \ldots$ are open subsets of X and $O = \bigcup_{i=1}^{\infty} O_i$,then

$$C_p(O) = \lim_{i \to \infty} C_p(O_i)$$

Proof. Monotonicity implies $\lim_{i\to\infty} C_p(O_i) \le C_p(O)$. To prove the opposite inequality, we may assume that $\lim_{i\to\infty} C_p(O_i) < \infty$. Let $\varepsilon > 0$ and $u_i \in A(O_i)$, $i = 1,2,3,\ldots$, and $g_{u_i} \in D(u_i) \cap L^p(X)$ be such that

$$\| u_i \|^p_{L^p(X)} + \| g_{u_i} \|^p_{L^p(X)} \le C_p(O_i) + \varepsilon$$

Now (u_i) is a bounded sequence in $L^p(X)$ and hence it has a weakly convergent subsequence, which we denote again (u_i).The sequence (g_{u_i}) is also bounded in $L^p(X)$ and hence, by passing to a subsequence, we may assume that $u_i \to u$ weakly in $L^p(X)$ and $g_{u_i} \to g$ weakly in $L^p(X)$ Using the Banach-Saks theorem we see that the sequence $v_j = j^{-1}\sum_{i=1}^{j} u_i$ converges to u in $L^p(X)$ and $g_{v_j} = j^{-1}\sum_{i=1}^{j} g_{v_j}$ converges to g in $L^p(X)$.Now there is a subsequence of (v_i) so that $v_j \to u$, μ-a.e. and $g_{v_k} \to g$, μ-a.e. The function u belongs to $W^{1,p}(X)$. On the other hand $v_j \to 1$ μ-a.e. in O and hence $u \ge 1$, $\boldsymbol{\mu}$-a.e. there. This means that $u \in A(O)$ By the weak lower semi continuity of norms

$$C_p(O) \le \| u \|^p_{L^p(X)} + \| g \|^p_{L^p(X)} \le \liminf_{i\to\infty}(\| u_i \|^p_{L^p(X)} + \| g_{u_i} \|^p_{L^p(X)}) \lim_{i\to\infty} C_p(O_i) + \varepsilon$$

from which the claim follows by letting $\boldsymbol{\varepsilon} \to 0$.

Theorem(3.4). For each Cauchy sequence of functions in $W^{1,p}(X) \cap C(X)$ there is a subsequence which converges point wise *p*-*q*.e. in X . Moreover, the convergence is uniform outside a set of arbitrary small *p*-capacity.

Proof. There is a sub sequence of (u_i), which we still denote by (u_i),such that .

$$\sum_{i=1}^{\infty} 2^{ip} \| u_i - u_{u+1} \|^p_{W^{1,p}(X)} < \infty$$

For $i = 1,2,3,\ldots$, denote $E_i = \{x \in X : |u_i(x) - u_{i+1}(x) > 2^{-j}\}$ and $F_j = \bigcup_{i=j}^{\infty} E_i$. By continuity $2^i (u_i - u_{i+1})$ is admissible for E_i, which implies

$$C_P(E_i) \le \sum_{i=j}^{\infty} 2^{ip} \| u_i - u_{u+1} \|^p_{W^{1,p}(X)}$$

and by subadditivity we obtain

$$C_p(F_j) \le \sum_{i=j}^{\infty} C_p(E_i) \le \sum_{i=j}^{\infty} 2^{ip} \| u_i - u_{u+1} \|^p_{W^{1,p}(X)}$$

Hence

$$C_p\left(\bigcap_{j=1}^{\infty} F_j\right) \le \lim_{j\to\infty} C_p(F_j) = 0$$

and (u_i) converges in . Mor $X \setminus \bigcap_{j=1}^{\infty} F_j$ eover,

$$|u_j - u_k| \le \sum_{i=j}^{k-1} |u_i - u_{i+1}| \le \sum_{i=j}^{k-1} 2^{-i} \le 2^{1-j}$$

in X / F_j for every $k > j$, which means that (u_i) convergence is uniform in X / F_j. The theorem follows. A function $u : X \to [-\infty,\infty]$ is p-quasi continuous in X if for every $\varepsilon > 0$ there is a set E such that $C_p(E) < \varepsilon$ and the restriction of u to $X \setminus E$ is continuous. By outer regularity , we may assume that E is open.By [8], $W^{1,p}(X)$ is a Banach space and by [8], $C(X) \cap W^{1,p}(X)$ is a dense subspace of $W^{1,p}(X)$ and hence completeness implies that $W^{1,p}(X)$ can be characterized as the completion of $C(X) \cap W^{1,p}(X)$ in the norm defined by (8). This means that $u \in W^{1,p}(X)$ if and only if there exist sequences of functions $u_i \in L^p(X) \cap C(X)$ and $g_i \in D(u_i - u)$ such that $u_i \to u$ and $g_i \to 0$ in $L^p(X)$ We deduce from the previous theorem that the limit function is p-quasi continuous and hence each Sobolev function,has a p-quasi continuous representative.

4.Capacity and measure:

We are mainly interested in the sets of vanishing capacity, since they are in some sense exceptional sets in the theory Sobolev spaces. Our first result is rather immediate.

Lemma (4.1). Let $\mu(E) \le C_p(E)$ for every $E \subset X$.

Proof. If $u \in A(E)$, then there is an open $O \supset E$ such that $u \ge 1$ in O and hence

$$\mu(E) \le \mu(O) \le \|u\|^p_{L^p(X)} \le \|u\|^p_{w^{1,p}(X)} .$$

We obtain the claim by taking the infimum over all $u \in A(E)$.

In particular, sets of capacity zero are also of measure zero. From now on we assume in this section that the measure is nontrivial, Borel regular and that there is $c_d \ge 1$ such that

$$\mu B(x, 2r) \le c_d \mu B(x, r) \tag{11}$$

for all $x \in X$ and $r > 0$. A measure satisfying the condition (11) is said to be doubling and the constant cd is called the doubling constant. If μ is doubling, then every open set has a nonzero measure. If $0 < r < R < \infty$ and $x \in X$, then iterating the doubling condition we get

$$\mu B(x, R) \le c \left(\frac{R}{r}\right)^s \mu(B(x, r)) \tag{12}$$

Where

$$s = \frac{\log c_d}{\log 2} \tag{13}$$

and c depends only on the doubling constant c_d . Observe that in the Euclidean case with the Lebesgue measure s equals to the dimension of the space. Hence (13) defines a dimension related to the doubling measure μ .

Theorem (4.2). Let $x_0 \in X$ and $0 < r \le 1$. If μ is doubling , then

$$C_p(B(x_0, r)) \le c r^{-p} \mu B(x_0, r) \tag{14}$$

where c depends only on the doubling constant and p.

proof. Let $0<r<R$ and define

$$u(x)=\begin{cases}\dfrac{R-d(x,x_0)}{R-r}, & x\in B(x_0,R)\setminus B(x_0,r)\\ 1, & x\in B(x_0,r)\\ 0, & x\in X\setminus B(x_0,R)\end{cases}$$

and

$$g(x)=\begin{cases}\dfrac{1}{R-r}, & x\in B(x_0,R)\\ 0, & x\in X\setminus B(x_0,R)\end{cases}$$

We show that $g\in D(u)$.Let first $x,y\in A=B(x_0,R)\setminus B(x_0,r)$. Then

$$|u(x(-u(y)|=\frac{|d(x,x_0)-d(y,x_0)|}{R-r}\le\frac{d(x,y)}{R-r}$$

Hence (8) follows in this case. Next let $x\in A$ and $y\in B(x_0, r)$. Now

$$|u(x(-u(y)|=1-u(x)\le\frac{d(x,x_0)-r}{R-r}$$

Since $d(x,x_0)\ge r>d(y,x_0)$ we have

$$d(x,x_0)-r\le d(x,x_0)-d(y,x_0)\le d(x,y)$$

and (8) follows. The case $y\in A$ and $x\in B(x_0, r)$ is completely analogous. If $x,y\in B(x_0,r)$ or $x, y\in X/B(x_0, R)$, then clearly (8) holds. For the remaining cases, let $y\in B(x_0, r)$ and $x\in X/B(x_0, R)$. Now

$$|u(x(-u(y)|=1=u(x)\le\frac{R-r}{R-r}\le\frac{d(x,y)}{R-r}$$

which implies (8). Finally, if $x\in A$ and $y\in X\setminus B(x_0, R)$, then

$$|u(x(-u(y)|=u(x)\le\frac{R-d(x,x_0)}{R-r}$$

and since $d(x,x_0)<R<d(y,x_0)$, we obtain

$$R-d(x,x_0)\le d(y,x_0)-d(x,x_0)\le d(x,y)$$

and (8) again holds. Thus $g\in D(u)$,u belongs to $A(B(x_0,r))$ and

$$C_p(B(x_0,r))\le\int_{B(x_0,R)}u^p d\mu+\int_{B(x_0,R)}g^p d\mu\le(1+(R-r)^{-p})\mu(B(x_0,R))$$

$$\le(1+(R-r)^{-p})\left(\frac{R}{r}\right)^s\mu(B(x_0,R))$$

This leads to (14) if we choose $R=2r$.To this end, we need a simple equality which holds for any Borel measure μ.

5.Hausdorff measures: We recall the definition of Hausdorff measures. $E\subset X$ and suppose that $h:[0,\infty)\to[0,\infty)$ is a non-decreasing function so that

$$\lim_{r\to 0}h(r)=h(0)=0$$

For $0<\delta\le\infty$ and $E\subset X$ we define

$$\mathrm{H}_{\delta}^{h}(E)=\inf\left\{\sum_{i=1}^{\infty}h(r_i):E\subset\bigcup_{i=1}^{\infty}B(x_i,r_i),r_i\le\delta\right\}$$

Now

$$\mathrm{H}^{h}(E)=\lim_{\delta\to0}\mathrm{H}_{\delta}^{h}(E)=\sup_{\delta>0}\mathrm{H}_{\delta}^{h}(E)$$

produces the standard (spherical) *h*-Hausdorff measure of *E*. If $h(t) = t^{s}$ for $0 \le s < \infty$, then we obtain the ***s***-dimensional (spherical) Hausdorff measure which we denote by $\mathcal{H}^{s}$. The Hausdorff dimension of a set $E \subset X$ is

$$\dim E=\inf\left\{s:\mathrm{H}_{\delta}^{h}(E)=0\right\}=\sup\left\{s:\mathrm{H}^{s}(E)=\infty\right\}$$

For the properties of Hausdorff measures we refer to [13].

We say that a measure μ is regular with dimension $s>0$, if there is $c\ge1$ such that

$$c^{-1}r^{s}\le\mu(B(x,r))\le cr^{s} \tag{15}$$

for each $x \in X$ and $0 < r \le \operatorname{diam}(X)$. If μ is regular with dimension *s***.** Moreover, *X* has Hausdorff dimension *s* and there is a constant $c > 0$ such that $c^{-1}\mathrm{H}^{s}(E)\le\mu(E)\le c\mathrm{H}^{s}(E)$ for every $E \subset X$.

Theorem (5.1). Let $\mathrm{h}:[0,\infty)\to[0,\infty)$,

$$h(t)=\begin{cases}t^{s-p} & for\ s>p\\ \left(\log\dfrac{1}{t}\right)^{1-s} & for\ s=p\end{cases}$$

If μ is regular with dimension ***s***, then for every $E\subset X$, $C_p(E)\le c\mathrm{H}^{h}(E)$ .The constant ***c*** depends only on *p* and the constant in (28)**.**

Proof. Let $B(x_i,r_i),i=1,2,3,....$ be any covering of *E* such that the radii satisfy $r_i\le\dfrac{1}{2}$.

Now (14) and (15) yield

$$C_p(B(x_i,r_i))\le\begin{cases}\left(\log\dfrac{1}{r_i}\right)^{1-s} & ,\ for\ s=p\\ cr_i^{s-p} & ,\ for\ s<p\end{cases}$$

and subadditivity implies

$$C_p(E)\le c\sum_{i=1}^{\infty}h(r_i)$$

The claim follows by taking the infimum over all coverings by balls and letting the radii tend to zero.

6. Invariant Sobolev-Lorentz Capacity on $\mathbb{R}^n$

We recall that for $1\le p<\infty$ and $0\le\lambda\le n$, the Morrey space $L^{P,\lambda}(\mathbb{R}^n)$is defined to be the linear space of measurable functions $u\in L^{I}_{loc}(\mathbb{R}^n)$ such that

$$\|u\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n}\sup_{r>0}\left(r^{-\lambda}\int_{B(x,y)}|u(y)|^p dy\right)^{1/p} < \infty$$

Definition (6.1). Let $1<p<\infty$ and $1\le q\le\infty$, $X = L^{p,q}(\Omega,\mathbb{R}^n)$. A function f in X is said to have absolutely continuous normin X if and only if $\|f_{\chi E_k}\|_X \to 0$ for every sequence E_k satisfying $E_k \to \phi$ a.e.

Theorem(6.2). Let $\Omega\subset\mathbb{R}^n$. Suppose $1<p<\infty$ and $1\le q\le\infty$. If $f\in L^{p,q}(\Omega)$ and $g\in L^{p,q}(\Omega)$, then

$$\int_\Omega |f(x)g(x)|\,dx \le \|f\|_{L^{p,q}(\Omega)}\|g\|_{L^{p',q'}(\Omega)}$$

Proof. We have to analyze two situations, depending on whether $q\in(1,\infty)$ or not. Suppose first that $1<p<\infty$. Then $1<q<\infty$ and by Hölder's inequality, we have

$$\int_0^\infty f^*(s)g^*(s)\,ds = \int_0^\infty f^*(s)s^{1/p-1/q}g^*(s)s^{1/p'-1/q'}\,ds \le \|f\|_{L^{p,q}(\Omega)}\|g\|_{L^{p',q'}(\Omega)}$$

By using this and [300], we get the desired conclusion for $1<q<\infty$.

We assume now without loss of generality that $q=1$. The case $q=\infty$ is similar. If $q=1$ then $q'=\infty$ and we have

$$\begin{aligned}\int_0^\infty f^*(s)g^*(s)\,ds &= \int_0^\infty f^*(s)s^{1/p-1}g^*(s)s^{1/p'}\,ds\\ &\le \sup_{s>0} g^*(s)s^{1/p'}\int_0^\infty f^*(s)s^{1/p-1}\,ds\\ &\le \|g\|_{L^{p',\infty}(\Omega)}\|f\|_{L^{p,1}(\Omega)}\end{aligned}$$

By using this and [1] we get the desired conclusion for $q=1$ as well. This finishes the proof.

As an application of Theorem (6.1) we have the following result

Proposition (6.3). Suppose $1<p<\infty$ and $1\le q\le\infty$. Then

(i) $\|Tf\|_{L^{p,q}(\mathbb{R}^n)} \le \|f\|_{L^{p,q}(\mathbb{R}^n)}$ for every $f\in C_0.(\mathbb{R}^n)$

(ii) If $1\le q\le p$, then $\|Tf\|_{L^{p,q}(\mathbb{R}^n)} \le \|f\|_{L^{p,q}(\mathbb{R}^n)}$ for every $f\in C_0(\mathbb{R}^n)$.

Proof. We fix $p\in(1,\infty)$ and $q\in[1,\infty]$ .Let $f\in C_0(\mathbb{R}^n)$. It is easy the discussion afterwards) that $Tf\in C_0(\mathbb{R}^n)$.

Let $g\in L^{p',q'}(\mathbb{R}^n)$ .Without loss of generality we can assume that g is supported in $\operatorname{supp} Tf$ .Then it follows from Theorem (6.2) that $g\in L^1(\mathbb{R}^n)$. Moreover, we have

$$\begin{aligned}\int_{\mathbb{R}^n}(Tf)(x)g(x)\,dx &\le \int_{\mathbb{R}^n}|(Tf)(x)g(x)|\,dx\\ &\le \int_{\mathbb{R}^n}\left(\int_{SO(n)}\left(|f(Hx)|\,d\mu(H)\right)\right)|g(x)|\,dx \qquad (16)\end{aligned}$$

$$\leq \int_{SO(n)} \left(\int_{\mathbb{R}^n} |f(Hx)g(x)|\,dx \right) d\mu(H)$$

where we used Fubini's theorem for the equality in the sequence. It is easy to see that

$|f \circ H|^* = |f|^*$ for every $H \in SO(n)$. Since μ is a probability measure, we obtain, via (16):

$$\int_{\mathbb{R}^n} |(Tf)(x)\,g(x)|\,dx \leq \int_0^\infty f^*(s)g^*(s)\,ds \tag{17}$$

From (17) it follows:

$$\int_0^\infty (Tf)^*(s)g^*(s)\,ds \leq \int_0^\infty f^*(s)g^* \tag{18}$$

By using (18) together with [1], we get the desired conclusion.

7.Basic properties of the n , q **relative capacity:**

Usually, a capacity is a monotone and subadditive set function. The following theorem will show, among other things, that this is true in the case of the n,q relative capacity. We follow[2].

Next we get some estimates for the n,q relative capacity of the spherical condenser $(\tilde{B}(0,r), B(0,1))$. Lower estimates for the n,q relative capacity. The lower estimates for the relative capacity are always harder to get than the upper estimates. However, we start with the lower ones.

Let $r \in (0,1)$.We define $\tilde{W}(\tilde{B}(0,r), B(0,1)) = T(W(\tilde{B}(0,r), B(0,1)))$.

Theorem (7.1). Let $1 \leq q < \infty$ be fixed and let q' be its Hölder conjugate. Then there exists a constant $C(n,q) > 0$ such that

$$C(n,q)^{-1}\left(\ln\frac{1}{r}\right)^{-n/q'} \leq cap_{n,q}(\tilde{B}(0,r), B(0,1)) \leq C(n,q)\left(\ln\frac{1}{r}\right)^{-n/q'} \text{ for every } 0 < r < e^{-1/(n-1)}$$

We examine the relationship between Hausdorff measures and the Sobolev-Lorentz *n, q*-capacity .

Theorem(7.2). (Poincare inequality for Sobolev- Lorentz spaces) .

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $1 \leq q < \infty$ be fixed. Then there exists a constant $C(n,q)$ such that

$$\|u\|_{L^{n,q}(\Omega)} \leq C(n,q)\,|\Omega|^{1/n}\,\|\nabla u\|_{L^{n,q}(\Omega,\mathbb{R}^n)} \tag{19}$$

for every $u \in C_0^\infty(\Omega)$.

$$|u(x)| \leq \frac{1}{w_{n-1}}(I_1|\nabla u|)(x) \tag{20}$$

for every $x \in \mathbb{R}^n$. We recall that for every measurable function f in $\mathbb{R}^n$, $I_1 f$ is its Riesz potential of order 1. An application of Hardy-Littlewood-Sobolev theorem of fractional integration together with Theorem (6.2), and (20) yields the desired conclusion.

8.Results :

Corollary (8.1). Let G be a measurable subset of $\mathbb{R}^n$, $n \geq 1$, having finite measure, and let $f_m \in L^1(G)$. Then

$$\min_{c_m \in \mathbb{R}} \int_G \sum |f_m(x) - c_m| dx = \int_G \sum \left| f_m(x) - f_m^{\circ}\left(\frac{|G|}{2}\right)\right| dx \qquad (21)$$

where f_m° is defined as in (3), with Q replaced by G .

Proof. We prove (21) under the assumption that f_m is nonnegative. The general case can be dealt with analogously, on splitting f_m into its positive and negative parts. The fact that the minimum in (21) is attained is an easy consequence of the dominated convergence theorem for the Lebesgue integral.

Now, since

$$\sum f_m(x) = \int_0^{\infty} \sum \chi_{\{f_m > t\}}(x) dt \text{ , for } x \in G$$

then, for every $c_m \in \mathbb{R}^n$,

$$\int_G \sum |f_m(x) - c_m| dx = \int_{\{f_m > c_m\}} \sum \left(\int_0^{\infty} \sum \chi_{\{f_m > t\}}(x) dt - c_m \right) dx$$

$$+ \int_{\{f_m > c_m\}} \sum \left(c - \int_0^{\infty} \chi_{\{f_m > t\}}(x) dt \right) dx$$

$$= \int_0^{\infty} \int_G \sum \chi_{\{f_m > t\}}(x) dx dt - c_m |f_n > c_m| + \sum c_m \left(|G| - |\{f_n > c_m\}| \right) - \int_0^{\infty} \int_G \sum \chi_{\{f_m < c_m\}}(x) dx dt$$

$$= \int_{c_m}^{\infty} \sum |\{f_m > t\}| dt - \int_0^{c_m} \sum |\{f_m > t\}| dt + c_m |G|$$

Thus, $\int_G \sum |f_m(x) - c_m| dx$ is a locally absolutely continuous function of c_m in $\mathbb{R}$ whose derivative equals to $|G| - 2|\{f_m > c_m\}|$ for a.e. $c_m \in \mathbb{R}$. Hence, (21) follows.of course, in general, $f_m^{\circ}(|G|/2)$ need not be the unique minimizer in (21).

Corollary (8.2). Let $X(Q)$ be an r.i. space. Then

$$\sum \frac{1}{s_m} \left\| r_m^{1/n} \chi_{(0,s)}(r_m) \right\|_{\tilde{X}(0,1)} \leq \sum s_m^{-1/n'} \|_{\tilde{X}(0,1)}, s_m \in \left(0, \frac{1}{2}\right) \qquad (22)$$

Proof. Given $s_m \in (0,1/2)$, we have

$$\sum \frac{1}{s_m} \left\| r_m^{1/n} \chi_{(0,s)}(r_m) \right\|_{\tilde{X}(0,1)} \leq \sum s_m^{-1/n'} \| \chi_{(0,1)}(r_m) \|_{\tilde{X}(0,1)}, s_m \in \left(0, \frac{1}{2}\right)$$

$$\leq 2^{-1/n'} \sum \| (s_m + r_m)^{-1/n'} \chi_{(0,s)}(r_m) \|_{\tilde{X}(0,1)}$$

$$\leq 2^{-1/n'} \sum \| (s_m + r_m)^{-1/n'} \chi_{(0,1-s)}(r_m) \|_{\tilde{X}(0,1)}$$

$$\leq 2^{-1/n'} \sum \| r_m^{-1/n'} \chi_{(s,1)}(r_m) \|_{\tilde{X}(0,1)}$$

Corollary(8.3). Let $\Omega \subset \mathbb{R}^n$. Suppose $0 \leq \varepsilon \leq \infty$. If $f_j \in L^{(1+\varepsilon),(1+2\varepsilon)}(\Omega)$ and $g_j \in L^{(1+\varepsilon),(1+2\varepsilon)}(\Omega)$,then

$$\int_{\Omega}\sum\left|f_j(x)g_j(x)\right|dx \le \sum\left\|f_j\right\|_{L^{(1+\varepsilon),(1+2\varepsilon)}(\Omega)}\left\|g_j\right\|_{L^{(1+\varepsilon)',(1+2\varepsilon)'}(\Omega)}$$

Proof. We have to analyze two situations, depending on whether $0<\varepsilon<\infty$ or not. Suppose first that $0<\varepsilon<\infty$. Then by Hölder's inequality, we have

$$\sum\int_0^{\infty} f_j^{*}(s)\,g_j^{*}(s)ds = \int_0^{\infty}\sum f^{*}_{\ j}(s)s^{1/\varepsilon/(1+\varepsilon)}\,g_j^{*}(s)ds\sum\left\|f_j\right\|_{L^{(1+\varepsilon),(1+2\varepsilon)}(\Omega)}\left\|g_j\right\|_{L^{(1+\varepsilon)',(1+2\varepsilon)'}(\Omega)}$$

By using this and [1], we get the desired conclusion for $0<\varepsilon<\infty$.

We assume now without loss of generality that $\varepsilon=0$. The case $\varepsilon=\infty$ is similar. If $\varepsilon=0$ then $\varepsilon'=\infty$ and we have

$$\begin{aligned}\sum\int_0^{\infty} f_j^{*}(s)\,g_j^{*}(s)ds &= \sum\int_0^{\infty} f_j^{*}(s)s^{1/\varepsilon}\,g_j^{*}(s)s^{1/(1+\varepsilon)'}ds\\ &\le \sup_{s>0}\sum g_j^{*}(s)s^{1/(1+\varepsilon)'}\int_0^{\infty} f_j^{*}(s)s^{1/\varepsilon}\,g_j^{*}(s)s^{1/\varepsilon}ds\\ &\le \sum\left\|f_j\right\|_{L^{(1+\varepsilon)',1}(\Omega)}\left\|g_j\right\|_{L^{(1+\varepsilon)',\infty}(\Omega)}\end{aligned}$$

By using this and [1] we get the desired conclusion for $\varepsilon=0$ as well. This finishes the proof.

Corollary (8.4). The function u^m belongs to $W^{1,p}(X)$ if and only if $u^m\in L^P(X)$ and there are functions $u_i^m\in L^P(X), i=1,2,3,\ldots,$ such that $u_i^m\to u^m$ μ-a.e. and $g_i^m\in D(u_i^m)\cap L^p(X)$ such that $g_i^m\to g^m$ μ-a.e. for some $g^m\in L^p(X)$.

Proof. If $u^m\in W^{1,P}(X)$,then the claim of the lemma is clear. To see the converse, suppose that $u^m, g^m\in L^P(X), g_i^m\in D(u_i^m)\cap L^p(X)$ and $u_i^m\to u^m$ μ-a.e. and $g_i^m\to g^m$ μ-a.e. Then

$$\left|u_i^m(x)-u_i^m(y)\right|\le d(x,y)(g_i^m(x)+g_i^m(y)) \qquad (23)$$

for all $x, y\in X\setminus F_i$ with $\mu(F_i)=0, i=1,2,\ldots$ Let $A\subset X$ be such that $u_i^m(x)\to u^m(y)$ and $g_i^m(x)\to g^m(x)$ for all $x\in X/A$ and $\mu(A)=0$. Write $F=A\cup S_{i=1}^{\infty}F_i$; then $\mu(F)=0$. Let $x, y\in X/F$, $x\ne y$. From (9) we obtain

$$\left|u^m(x)-u^m(y)\right|\le d(x,y)(g^m(x)+g^m(y)) \qquad (24)$$

and thus $g^m\in D(u^m)\cap L^p(X)$. This completes the proof.

Corollary (8.5). If $u^m\in W^{1,P}(X)$ and $E\subset X$ is μ-measurable with $0<\mu(E)<\infty$, then for every

$$\int_E\sum\left|u^m-u_E^m\right|^p d\mu \le 2^p\,diam(E)\int_E g^{m\,p}\,d\mu$$

Proof. By the Hölder inequality

$$\sum\left|u^m(x)-u_E^m\right|\le\fint_E\sum\left|u^m(x)-u^m(y)\right|d\mu(y)\le\sum\left(\fint_E\sum\left|u^m(x)-u^m(y)\right|^p d\mu(y)\right)^{\frac{1}{p}}$$

An integration completes the proof, because

$$\int \sum \left|u^m(x)-u^m(y)\right|^p d\mu(y) \le \int_E \int_E \sum \left|u^m(x)-u^m(y)\right|^p d\mu(y)$$

$$\le 2^{p-1} diam(E)^p \int_E \sum \left(g^m(x)^p + \int_E g^m(y)^p d\mu(y) \right) d\mu(x)$$

$$\le 2^{p-1} diam(E)^p \int_E \sum g^{m\,p} d\mu(x)$$

Corollary(8.6). Suppose $0 \le \varepsilon \le \infty$. Then

(i) $\sum \left\| T^j f_j \right\|_{L^{((1+\varepsilon),(1+\varepsilon))}(\Omega,\mathbb{R}^n)} \le \sum \left\| f_j \right\|_{L^{((1+\varepsilon),(1+\varepsilon))}(\Omega,\mathbb{R}^n)}$ for every $f_j \in C_0(\mathbb{R}^n)$.

(ii) If $\varepsilon \ge 0$, then $\sum \left\| T^j f_j \right\|_{L^{(p,q)}(\mathbb{R}^n)} \le \sum \left\| f_j \right\|_{L^{(p,q)}(\mathbb{R}^n)}$ for every $f_j \in C_0(\mathbb{R}^n)$.

Proof. We fix $0 \le \varepsilon \le \infty$.Let $f_j \in C_0(\mathbb{R}^n)$. It is easy the discussion afterwards) that $T^j f_j \in C_0(\mathbb{R}^n)$.

Let $g_j \in L^{(1+2\varepsilon)'(1+\varepsilon)'}(\mathbb{R}^n)$ .without loss of generality we can assume that g is supported in supp$T^j f_j$ .Then it follows from Theorem (6.2) that $g_j \in L^1(\mathbb{R}^n)$. Moreover, we have

$$\int_{R^n} \sum \left|(T^j f_j)(x) g_j(x)\right| dx \le \int_{R^n} \sum \left|(T^j f_j(x) g_j(x)\right| dx$$

$$\le \int_{\mathbb{R}^n} \sum \left(\int_{SO(n)} \left(\left| f_j(Hx) \right| d\mu(H) \right) \right) |g(x)| dx \qquad (25)$$

$$\le \int_{SO(n)} \sum \left(\int_{\mathbb{R}^n} \left(\left| f_j(Hx) \right| d\mu(H) \right) \right) |g(x)| d\mu(H)$$

where we used Fubini's theorem for the equality in the sequence. It is easy to see that $\left| f_j \circ H \right|^* = \left| f_j \right|$ for every $H \in SO(\mathrm{n})$. Since μ is a probability measure, we obtain, via (25):

$$\int_{R^n} \sum \left|(T^j f_j(x) g_j(x)\right| dx \le \int_0^\infty \sum f_j^*(s) g_j^*(s) ds \qquad (26)$$

From (26) it follows:

$$\int_0^\infty \sum \left(T^j f_j^*\right)(s) g_j^*(s) ds \le \int_0^\infty \sum f_j^* g_j^*(s) ds \qquad (27)$$

By using (27) together with [1], we get the desired conclusion.

Acknowledgment: I wish to thank the referee for his / her useful comments and suggestions which will improve the final version of this paper **.**

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