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Homotopy Perturbation Method for Application Viscous Flows in Cylindrical Polar Coordinates

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Abstract

In this paper, the Naveir-Stokes equation in cylindrical coordinates was used as the approximate solution. To get the analytical approximation the Homotopy perturbation method has been used. Depending upon the different available options for the linear operator, we also have the benefit of choosing different initial approximations to start our analysis. The analysis is accomplished without computing the Adomian's polynomials.

Keywords: ***Homotopy perturbation method, Adomian's polynomials, Viscous, Cylindrical Polar Coordinates.***

1. Introduction

In physical processes, the Time-delay Burgers equation is widely used and does not have a precise analytical solution. To solve this equation the approximate methods are used. Fahmy [1-3] solved the generalized time delay convective Fishers equation, the generalized time delay Burgers-Huxley equation, and the generalized time delay Burgers--Fisher equation by using the factorization method to find explicit particular traveling wave solutions and yields to the general solution. The tanh function method was used by Fahmy et al [2] to construct exact multiple solutions and triangular periodic solutions and calculated the numerical solutions of this equation by using the Adomian decomposition method and variational iteration

method. Kim and Sakthivel [6] used the $\frac{G'}{G}$ expansion method, which is implemented to define traveling wave solutions for the time-delay Burgers-Fisher equation. The trigonometric functions and hyperbolic functions are used to express these solutions. And that by using the first –integral which is based on the ring theory of commutative algebra.

Ji-Huan He [8,9,10] proposed the Homotopy perturbation method, to improve this method to solve various nonlinear problems many authors tried [4,7,11,12,13,15]. HPM yields a very prompt convergence of the solution series and sometimes one iteration leads to high accuracy of the solution.

2. Analysis

Consider the governing equations of motion for the two dimensional flow field for a viscous fluid in a tube which are described by the cylindrical coordinate transformation of the Navier-Stokes equations read as;

$$u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}=\nu\left(\frac{\partial u}{\partial r^{2}}^{2}+\frac{1}{r}\frac{\partial^{2}u}{\partial r}+\frac{\partial^{2}u}{\partial z^{2}}\right)-\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{1}$$

$$u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}=\nu\left(\frac{\partial^{2}u}{\partial r^{2}}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v^{2}}{r^{2}}+\frac{\partial^{2}v}{\partial z^{2}}\right)-\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{2}$$

It is suggested that the rotational motion of the fluid is negligible. Then the equation of continuity reads is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv)+\frac{\partial u}{\partial z}=0 \tag{3}$$

where u is fluid velocity components in the axial x coordinate and v is in the radial coordinate r , and the fluid pressure is described by P, the fluid density by ρ, and the kinematic viscosity by v for the fluid. Introducing and labeling the stream function as Φ, then we may have,

$$u = -\frac{1}{r}\frac{\partial \Phi}{\partial r}, \quad \text{and} \quad v = -\frac{1}{r}\frac{\partial \Phi}{\partial z}, \tag{4}$$

The equation of continuity is satisfied identically. The dynamical equation of motion in term of the stream function Φ are obtained by eliminating P between (2) and (3),), and making us of the relation (4), it is read as;and the Jacobean defined as $\Im$ as a linear operator which is defined as;

$$\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\phi\Im\Phi}{\partial\Phi} = v\Im^2\Phi \tag{5}$$

$$\Im = \frac{\partial^2}{\partial r^2} - \left(\frac{1}{r}\right)\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{6}$$

and the Jacobean defined as

$$\frac{\partial(L\Phi,\Phi)}{\partial(r,z)} = \frac{\partial(\Im\Phi)}{\partial r}\frac{\partial\Phi}{\partial z} - \frac{\partial\Phi}{\partial r}\frac{\partial(\Im\Phi)}{\partial z} = \begin{vmatrix} \frac{\partial(\Im\Phi)}{\partial r} & \frac{\partial\Phi}{\partial r} \\ \frac{\partial(\Im\Phi)}{\partial z} & \frac{\partial\Phi}{\partial z} \end{vmatrix} \tag{7}$$

Now, we here mainly discuss to forms of the linear operator $\Im$ defined by equation (5). We will split the linear operator $\Im$ in two parts and discuss the two cases. It is to note that in the Homotopy perturbation method we are free to choose the linear operator. This mainly depends upon the given form of the initial or boundary condition and the problem under investigation Therefore, depending upon our choices and the possibilities for the appearance of the auxiliary linear

form of operator in the problem we consider two cases here.

State 1:

The first form of the linear operator extracted from equation (5) for the possible form of the linear operator is supposed to be;

$$\Im_1 = \frac{\partial^2}{\partial r^2} - \left(\frac{1}{r}\right)\frac{\partial}{\partial r} \tag{8}$$

Then the operator $\Im$ becomes $\Im = \Im_1 + \frac{\partial^2}{\partial z^2}$, which implies that

$\Im^2 = \Im_1^2 + 2\frac{\partial^2}{\partial z^2}\Im_1 + \Im_1{}^2$ then

$$\Im^2\phi = \Im_1^2\Phi + 2\frac{\partial^2\Phi}{\partial z^2}\Im_1\phi + \Im_1{}^2\Phi \tag{9}$$

Using (9) in (5), the equation (5) takes the following form

$$\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial^2\Phi}{\partial z^2}\Im\Phi = \nu\left(\Im_1^2\Phi + 2\frac{\partial^2}{\partial z^2}(\Im_1^2\Phi) + \frac{\partial^4\Phi}{\partial z^4}\right)$$

taking ν^{-1} both sides

$$\Im_1^2\Phi + 2\frac{\partial^2(\Im_1\Phi)}{\partial z^2} + \frac{\partial^4\Phi}{\partial z^4} = \nu^{-1}\left(\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\Phi}{\partial z}\Im\Phi\right)$$

In order to apply the proposed Homotopy perturbation method to the given problem, we need to define the nonlinear term appearing in the governing equations. Therefore, we define the nonlinear term as $N\Phi$ in the above equation which is given as;

$$N\Phi = \frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\Phi}{\partial z}\Im\Phi$$

Then we get the following nonlinear form of equation for our analysis

$$\mathfrak{I}_1^2\Phi = \left(\nu^{-1}\left(\frac{1}{r}\frac{\partial(\mathfrak{I}\Phi,\Phi)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\Phi}{\partial z}\mathfrak{I}\Phi \right) - 2\frac{\partial^2(\mathfrak{I}_1\Phi)}{\partial z^2} - \frac{\partial^4\Phi}{\partial z^4} \right)$$

Operating $\mathfrak{I}_1^{-1}$ on both sides of the above equation,

$$\mathfrak{I}_1\Phi = \mathfrak{I}_1^{-1}\left(\nu^{-1}\left(\frac{1}{r}\frac{\partial(\mathfrak{I}\Phi,\Phi)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\Phi}{\partial z}.\mathfrak{I}\Phi \right) - 2\frac{\partial^2(\mathfrak{I}_1\Phi)}{\partial z^2} - \frac{\partial^4\Phi}{\partial z^4} \right) \tag{10}$$

Using Homotopy perturbation method (HPM) proposed by J. H. He (He, 2006), we construct a Homotopy for equation (10) as; $\omega(\mathrm{r},\mathrm{Z};\sigma)\colon \Omega\times[0,1]\to\Re$. This satisfies $H(\nu,\sigma) = (1-\sigma)[\ \mathfrak{I}(v) - \mathfrak{I}(u_0)] + \big(\mathrm{A}(v) - \mathrm{f}(\mathrm{r})\big) = 0,$ and her $\sigma\epsilon[0,1]$ is designed to be an embedding parameter,

$$\begin{aligned}(1-\sigma)H(\omega\sigma) &= (\mathfrak{I}_1\omega - \mathfrak{I}_1\omega_0) \\ &-\sigma\left[\nu^{-1}\mathfrak{I}_1^{-1}\left(\frac{1}{r}\frac{\partial(\mathfrak{I}w,w)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial w}{\partial z}.\mathfrak{I}\Phi \right) - 2\mathfrak{I}_1^{-1}\frac{\partial(\mathfrak{I}_1\Phi)}{\partial z^2} - \mathfrak{I}_1^{-1}\frac{\partial^4\Phi}{\partial z^4} \right] = 0.\end{aligned} \tag{11}$$

Suppose the solution of (11) is of the form of

$$\omega(r,z,\sigma) = \omega_0 + \sigma\omega_1 + \sigma^2\omega_2 + \ldots\ldots\ldots \tag{12}$$

Using (12) in (11) we get

$$\begin{aligned}&\mathfrak{I}_1(\omega_0 + \sigma\omega_1 + \sigma^2\omega_2 + \ldots\ldots\ldots) - \mathfrak{I}_1\phi_1 + \sigma\mathfrak{I}_1 w_0 - \sigma\nu^{-1}\mathfrak{I}_1^{-1}\left[\frac{1}{r}\frac{\partial\mathfrak{I}(\omega_0+\sigma\omega_1+\ldots,\omega_0+\sigma\omega_1+\ldots)}{\partial(r,z)} \right] \\ &+\sigma\nu^{-1}\mathfrak{I}_1^{-1}\left[\frac{2}{r}\frac{\partial(\omega_0+\sigma^1\omega_1+\ldots,)}{\partial z}.\mathfrak{I}(\omega_0+\sigma^1\omega_1+\ldots,) \right] + \\ &\sigma^{-1}\mathfrak{I}_1^{-1}\left[-2\frac{\partial^2\mathfrak{I}_1(\omega_0+\sigma\omega_1+\ldots,)}{\partial z^2} - \frac{\partial^4\mathfrak{I}_1(\omega_0+\sigma\omega_1+\ldots,)}{\partial z^4} \right] = 0\end{aligned} \tag{13}$$

Now we simplify the quantities enclosed in brackets,

$$\frac{\partial(\Im(\omega_0+\sigma\omega_1+...),(\omega_0+\sigma\omega_1+...))}{\partial(\mathbf{r},z)}=\begin{vmatrix}\frac{\partial(\Im\omega_0+\sigma\Im\omega_1+...)}{\partial r} & \frac{\partial(\omega_0+\sigma\omega_1+...)}{\partial r}\\ \frac{\partial(\Im\omega_0+\sigma\Im\omega_1+...)}{\partial z} & \frac{\partial(\omega_0+\sigma\omega_1+...)}{\partial z}\end{vmatrix}$$

$$=\frac{\partial(\Im\omega_0+\sigma\Im\omega_1+\sigma^2\Im\omega_2+...)}{\partial r}.\frac{\partial(\Im\omega_2,\omega_0)}{\partial(r,z)}-\frac{\partial(\Im\omega_0+\sigma\Im\omega_1+\sigma^2\Im\omega_2+...)}{\partial z}.\frac{\partial(\omega_0+\sigma\omega_1+\sigma^2\omega_2+...)}{\partial r} \quad (14)$$

$$+\sigma^2\left(\left((\frac{\partial\Im\omega_2}{\partial r}.\frac{\partial\omega_0}{\partial z}-\frac{\partial\Im\omega_2}{\partial z}.\frac{\partial\omega_0}{\partial r})+(\frac{\partial\Im\omega_0}{\partial r}.\frac{\partial\omega_2}{\partial z}-\frac{\partial\Im\omega_0}{\partial z}.\frac{\partial\omega_2}{\partial r})+(\frac{\partial\Im\omega_1}{\partial r}.\frac{\partial\omega_1}{\partial z}-\frac{\partial\Im\omega_1}{\partial z}.\frac{\partial\omega_1}{\partial r})\right)\right)+O(\sigma^3),$$

$$=\sigma^0\left(\frac{\partial(\Im\omega_0,\omega_0)}{\partial(r,z)}\right)+\sigma^1\left(\frac{\partial(\Im\omega_1,\omega_0)}{\partial(r,z)}+\frac{\partial(\Im\omega_0,\omega_1)}{\partial(r,z)}\right)+\sigma^2\left(\frac{\partial(\Im\omega_2,\omega_0)}{\partial(r,z)}+\frac{\partial(\Im\omega_0,\omega_2)}{\partial(r,z)}+\frac{\partial(\Im\omega_1,\omega_1)}{\partial(r,z)}\right)+...$$

The calculations made in (14) is according to the definition of the Jacobean, and

$$\frac{\partial(\omega_0+\sigma^1\omega_1+...)}{\partial z}\cdot(\Im(\omega_0+\sigma^1\omega_1+...))=\sigma^0\left(\Im\omega_0\frac{\partial\omega_0}{\partial z}\right)+\sigma^1\left(\Im\omega_1\frac{\partial\omega_0}{\partial z}+\Im\omega_0\frac{\partial\omega_1}{\partial z}\right) \quad (15)$$

$$+\sigma^2\left(\Im\omega_2\frac{\partial\omega_0}{\partial z}+\Im\omega_1\frac{\partial\omega_1}{\partial z}+\Im\omega_0\frac{\partial\omega_2}{\partial z}\right)+.....,$$

(15) Combining the terms containing the equal powers of σ in equation (14) and (15)

$$c_1=\sigma^0\left(\frac{1}{r}\left(\frac{\partial(\Im\omega_0,\omega_0)}{\partial(r,z)}\right)-\Im_1^{-1}\left(\frac{-2}{r^2}\frac{\partial\omega_0}{\partial z}\Im\omega_0\right)\right), \quad (15.a)$$

$$c_1=\sigma^0\left(\left(\frac{\partial(\Im\omega_1,\omega_0)}{\partial(r,z)}+\frac{\partial(\Im\omega_0,\omega_1)}{\partial(r,z)}\right)-\left(\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\cdot\Im\omega_1+\frac{2}{r^2}\frac{\partial\omega_1}{\partial z}\cdot\Im\omega_0\right)\right), \quad (15.b)$$

$$c_1=\sigma^2\left(\left(\frac{\partial(\Im\omega_2,\omega_0)}{\partial(r,z)}+\frac{\partial(\Im\omega_0,\omega_2)}{\partial(r,z)}+\frac{\partial(\Im\omega_1,\omega_1)}{\partial(r,z)}\right)-\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\cdot\Im\omega_2-\frac{2}{r^2}\frac{\partial\omega_1}{\partial z}\cdot\Im\omega_1-\frac{2}{r^2}\frac{\partial\omega_2}{\partial z}\cdot\Im\omega_2\right), \quad (15.c)$$

and so on. now

$$\frac{\partial^2(\Im(\omega_0+\sigma^1\omega_1+\sigma^2\omega_2+...))}{\partial z^2}=\sigma^0\left(\frac{\partial^2\Im\omega_0}{\partial z^2}\right)+\sigma^1\left(\frac{\partial^2\Im\omega_1}{\partial z^2}\right)+\sigma^2\left(\frac{\partial^2\Im\omega_2}{\partial z^2}\right)+.....,$$

(16)

$$\frac{\partial^4(\Im(\omega_0+\sigma^1\omega_1+\sigma^2\omega_2+\ldots))}{\partial z^2}=\sigma^0\left(\frac{\partial^4\Im\omega_0}{\partial z^4}\right)+\sigma^1\left(\frac{\partial^4\Im\omega_1}{\partial z^4}\right)+\sigma^2\left(\frac{\partial^4\Im\omega_2}{\partial z^4}\right)+\ldots\ldots, \tag{17}$$

Now complaining the term containing the equal powers of σ in equations (16) and (17)

$$D_1=\left(-2\frac{\partial^2\Im_1\omega_0}{\partial z^2}\right)-\left(\frac{\partial^4\omega_0}{\partial z^4}\right) \tag{17.a}$$

$$D_2=\left(-2\frac{\partial^2\Im_1\omega_1}{\partial z^2}\right)-\left(\frac{\partial^4\omega_1}{\partial z^4}\right) \tag{17.b}$$

$$D_3=\left(-2\frac{\partial^2\Im_1\omega_2}{\partial z^2}\right)-\left(\frac{\partial^4\omega_2}{\partial z^4}\right) \tag{17.c}$$

and so on. Using equations (15.a), (15.b), (15.c), (17.a), (17.b), (5.71.c) in equation (13). We get from equation (13),

$$\begin{aligned}&(\sigma^0\Im_1\omega_0+\sigma^1\Im_1\omega_1+....)-\Im_1\Phi_0+\sigma^1\Im_1\omega_0-\\&\sigma\nu^{-1}\Im_1^{-1}(\sigma^0C_1+\sigma^1C_2+\sigma^2C_3+\ldots+\sigma^0D_1+\sigma^1D_2+\sigma^2D_3+\ldots)=0\end{aligned} \tag{18}$$

(18) Equating the coefficients of equal powers of σ we have the zeroth order problem as:

Zeroth Order problem: $\Im_1\omega_0-\Im_1\phi_0=0$, which implies $\omega_0=\Phi_0(r,z)$. (19)

Here Φ_0 is defined as the solution of homogenous equation $\Im_1^2\Phi_0=0$, (20)

subject to the pre-prescribed boundary conditions. Now to find the approximation for ω_1 for which we first find the inverse operator $\Im_1^{-2}$ and for it we consider equation (13), $\Im_1\Phi=0$. (21)

We now define $\Im_{rr} = \frac{\partial^2}{\partial r^2}$ & $\frac{\partial}{\partial r}$, takes the following form as $\Im_1 = \Im_{rr} - \frac{1}{r}\Im_r$,

$\Im_1\Phi = \Im_{rr}\Phi - \frac{1}{r}\Im_r\Phi$. Using equation (21) we get $\Im_{rr}\Phi - \frac{1}{r}\Im_r\Phi = 0$. Solving for $\Im_{rr}$ and $\Im_r$ that is for linear terms

$$\Im_{rr}\Phi = \frac{1}{r}\Im_r\Phi, \tag{22}$$

$$\Im_r\Phi = r\Im_{rr}\Phi, \tag{23}$$

Operating $\Im_r^{-1}$ on (24) and $\Im_{rr}^{-1}$ on (5.78) we get

$$\Phi = \varphi_1 + \Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\Phi\right), \tag{24}$$

$$\Phi = \varphi_2 + \Im_r^{-1}\left(r\Im_{rr}\Phi\right), \tag{25}$$

$\varphi_1$and φ_2 are the solutions of two homogenous equations $\Im_{rr}\Phi = 0$ and $\Im_r\Phi = 0$, respectively. The inverse linear operators $\Im_{rr}^{-1}$ and $\Im_r^{-1}$ are defined as

$$\begin{aligned}\Im_{rr}^{-1} &= \iint(.)drdr \\ \Im_r^{-1} &= \int(.)dr\end{aligned} \tag{26}$$

Adding (25) and (26) we get $2\Phi = \varphi_1 + \varphi_2 + \Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\Phi\right) + \Im_r^{-1}\left(r\Im_{rr}\Phi\right)$, and dividing both sides by 2, to get

$$\Phi = \frac{\varphi_1 + \varphi_2}{2} + \left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\Phi\right) + \Im_r^{-1}\left(r\Im_{rr}\Phi\right)\right], \tag{27}$$

$$\Phi = \varphi_9 + \frac{1}{2}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\Phi\right) + \Im_r^{-1}\left(r\Im_{rr}\Phi\right)\right] \tag{28}$$

Where $\Phi_0 = \frac{\varphi_1 + \varphi_2}{2}$ then

$$\begin{aligned}
\Phi_1 &= \frac{1}{2}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r^{-1}\left(r\Im_{rr}\right)\right]\Phi_0 \\
\Phi_2 &= \frac{1}{2}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r^{-1}\left(r\Im_{rr}\right)\right]\Phi_1 \\
&\vdots = \vdots \\
\Phi_{n+1} &= \frac{1}{2}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r^{-1}\left(r\Im_{rr}\right)\right]\Phi_n
\end{aligned}, \qquad (29)$$

Let the quantity in brackets is denoted by β

$$\left[\begin{aligned}
\Phi_1 &= \frac{1}{2}\beta\Phi_0 \\
\Phi_2 &= \frac{1}{2}\beta\Phi_1 = \frac{1}{2}\beta\frac{1}{2}\beta\Phi_0 = \frac{1}{2^2}\beta^2\Phi_0 \\
&\vdots = \vdots \\
\Phi_{n+1} &= \frac{1}{2^{n+1}}\beta^{n+1}\Phi_0
\end{aligned}\right] \qquad (30)$$

$$\Phi = \sum_{n=0}^{\infty}\frac{1}{2^n}\beta^n\Phi_0 = \sum_{n=0}^{\infty}\frac{1}{2^n}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r^{-1}(r\Im_{rr})\right]^n\Phi_0. \qquad (31)$$

Now the inverse of the linear operator is defined $\Im_r^{-2} = \left(\Im_{rr} - \frac{1}{r}\Im_r\right)^{-2}$ as

$$\Im_r^{-2} = \sum_{n=0}^{\infty}\frac{1}{2^n}\left[\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r\right) + \Im_r^{-1}(r\Im_{rr})\right]^n \qquad (32)$$

Now we come to equation (19) and define the zeroth order problem as, $\omega_0 = \Phi_0$.

And the 1st order problem as

$$\Im_1\omega_1 + \Im_1\omega_0 = \Im_1^{-1}(\nu^{-1}C_1 + D_1).$$

Substituting values from equation (15.a) (17.a) we get,

$$\Im_1\omega_1 + \Im_1\omega_0 = \Im_1^{-1}\left[\nu^{-1}\left(\frac{1}{r}\frac{\partial(\Im\omega_0,\omega_0)}{\partial(r,z)} - \Im\omega_0\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - 2\frac{\partial^2\Im_1\omega_0}{\partial^2 z} - \frac{\partial^4\omega_0}{\partial z^4}\right]$$

Operating $\Im_1$ on both sides of the above equation yields,

$$\Im_1^2\omega_1 + \Im_1^2\omega_0 = \nu^{-1}\left(\frac{1}{r}\frac{\partial(\Im\omega_0,\omega_0)}{\partial(r,z)} - \Im\omega_0\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - 2\frac{\partial^2\Im_1\omega_0}{\partial^2 z} - \frac{\partial^4\omega_0}{\partial z^4}$$

Making use of $\omega_0 = \Phi_0$, for the initial guess of HPM methodology, $\Im_1^2\Phi_0 = 0$,

$$\Im_1^2\omega_1 = \nu^{-1}\left(\frac{1}{r}\frac{\partial(\Im\omega_0,\omega_0)}{\partial(r,z)} - \Im\omega_0\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - 2\frac{\partial^2\Im_1\omega_0}{\partial^2 z} - \frac{\partial^4\omega_0}{\partial z^4}$$

Operating with $\Im_1^{-2}$ on both sides of the above expression,

$$\omega_1 = \nu^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial(\Im\omega_0,\omega_0)}{\partial(r,z)} - \Im\omega_0\frac{2}{r^2}\frac{\partial\omega_0}{\partial z}\right) - 2\Im_1^{-2}\frac{\partial^2\Im_1\omega_0}{\partial^2 z} - \Im_1^{-2}\frac{\partial^4\omega_0}{\partial z^4} \qquad (33)$$

Where $\Im_1^{-2}$ is given in equation (32). The 2nd order problem is given as, $\Im_1\omega_2 = \Im_1^{-1}(\nu^{-1}c_2 + D_2)$. Using the values of c_2 and D_2 from equations (15.b) and (17.b), we get

Operating with $\Im_1^{-1}$ both sides of the above equation to get,

$$\Im_1^{-1}\left(\nu^{-1}\left(\frac{1}{r}\left(\frac{\partial(\Im\omega_1,\omega_0)}{\partial(r,z)} + \frac{\partial(\Im\omega_0,\omega_1)}{\partial(r,z)}\right)\right) - \left(\Im\omega_1\frac{2}{r}\frac{\partial\omega_0}{\partial z} + \Im\omega_1\frac{2}{r^2}\frac{\partial\omega_1}{\partial z}\right) - \left(2\frac{\partial^2\Im_1\omega_1}{\partial z^2}\right) - \left(\frac{\partial^4\omega_0}{\partial z^4}\right)\right)$$

$$\omega_2 = \nu^{-1}\Im_1^{-2}\left(\left(\frac{1}{r}\frac{\partial(\Im\Phi_1,\Phi_0)}{\partial(r,z)} - \Im\Phi_1\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) + \left(\frac{1}{r}\frac{\partial(\Im\Phi_0,\Phi_1)}{\partial(r,z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\right)\right) - 2\Im_1^{-2}\frac{\partial^2\Im_1\omega_1}{\partial z^2} - \Im_1^{-2}\frac{\partial^4\omega_0}{\partial z^4} + \ldots, \quad (34)$$

Now since in the methodology of HPM, we suppose the following expression for the approximate solution of the problem

$$\Phi = \lim\omega = \omega_0 + \omega_1 + \omega_2 + \omega_3 + \ldots. \quad (35)$$

where the components of the series solution are defined to be as; $\omega_0 = \Phi_0$

$$\Phi_1 = \nu^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial(\Im\Phi_0,\Phi_0)}{\partial(r,z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) - 2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_0}{\partial^2 z} - \Im_1^{-2}\frac{\partial^4\Phi_0}{\partial z^4}$$

$$\Phi_2 = \nu^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial(\Im\Phi_1,\Phi_0)}{\partial(r,z)} - \Im\Phi_1\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) + \left(\frac{1}{r}\frac{\partial(\Im\Phi_0,\Phi_1)}{\partial(r,z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\right) - 2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_1}{\partial^2 z} - \Im_1^{-2}\frac{\partial^4\Phi_1}{\partial z^4}$$

and so on. If once Φ_0 is obtained, which can be easily obtained by constructing the Homotopy for the given problem and equating the coefficients for the zeroth order problem, then we can find Φ_1 terms of Φ_0 and in the similar fashion Φ_2 can be evaluated in terms of Φ_1 and Φ_2 The other higher order components can be easily obtained having the all other lower order values. Thus all the components of Φ can be calculated. The series solution thus $\Phi = \sum_{n=0}^{\infty}\Phi_n$, can be given the following form

$$\Phi = \Phi_0 \nu^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial(\Im\Phi_0,\Phi_0)}{\partial(r,z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) - 2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_0}{\partial z^2} - \Im_1^{-2}\frac{\partial^4\Phi_0}{\partial z^4}$$
$$+\nu^{-1}\Im_1^{-2}\left(\frac{1}{r}\frac{\partial(\Im\Phi_1,\Phi_0)}{\partial(r,z)} - \Im\Phi_1\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) + \left(\frac{1}{r}\frac{\partial(\Im\Phi_0,\Phi_1)}{\partial(r,z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\right) \quad (36)$$
$$-2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_1}{\partial z^2} - \Im_1^{-2}\frac{\partial^4\Phi_1}{\partial z^4} + \ldots\ldots$$

State 2:

We may have the other available or chosen form of the linear operator as

$\Im = \Im_{rr} - \frac{1}{\mathrm{r}}\Im_r + \Im_{zz}$ where as $\Im_r = \frac{\partial}{\partial \mathrm{r}}$, $\Im_{rr} = \frac{\partial^2}{\partial r^2}$ and $\Im_{zz} = \frac{\partial^2}{\partial z^2}$,

then from equation (5.59) we have

$$\nu\Im^2\Phi = \frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,z)} - \frac{2}{r^2}\frac{\partial\Phi\Im\Phi}{\partial\Phi}$$

taking ν^{-1} both sides of the above expression $\Im^2\Phi = \nu^{-1}\left(\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,z)} - \Im\Phi\frac{2}{r^2}\frac{\partial\Phi}{\partial Z}\right)$

$$\Im^2\Phi - \nu^{-1} = \left(\frac{1}{r}\frac{\partial(\Im\Phi,\Phi)}{\partial(r,z)} - \Im\Phi\frac{2}{r^2}\frac{\partial\Phi}{\partial Z}\right) = 0 \quad (37)$$

Using the methodology of HPM, we may construct a Homotopy for equation (37) as

$$\overline{\omega}(r,z;q):\Omega[0,1]\to\Re\cdot H(\overline{\omega},q) = (1-q)(\Im\overline{\omega} - \Im\Phi_0) - q\Im^{-1}\left(\nu^{-1}\left(\frac{1}{r}\frac{\partial(\Im\overline{\omega},\overline{\omega})}{\partial(r,z)} - \Im\overline{\omega}\frac{2}{r^2}\frac{\partial\overline{\omega}}{\partial Z}\right)\right) \quad (38)$$

Suppose the solution for equation (38) is of the following form

$$\overline{\omega}(r,z;q) = q^0\overline{\omega}_0(r,z) + q^1\overline{\omega}_1(r,z) + q^2\overline{\omega}_2(r,z) + \ldots\ldots, \quad (39)$$

Where as

$$\frac{\partial(\Im\overline{\omega},\overline{\omega})}{\partial(r,z)}=\begin{vmatrix}\frac{\partial\Im(\overline{\omega}_0+q\overline{\omega}_1+\ldots)}{\partial r} & \frac{\partial(\overline{\omega}_0+q\overline{\omega}_1+\ldots)}{\partial r}\\ \frac{\partial\Im(\overline{\omega}_0+q\overline{\omega}_1+\ldots)}{\partial z} & \frac{\partial(\overline{\omega}_0+q\overline{\omega}_1+\ldots)}{\partial z}\end{vmatrix}$$
$$=\frac{\partial\Im(\overline{\omega}_0+q\overline{\omega}_1+\ldots)}{\partial r}\cdot\frac{\partial(\overline{\omega}_0+q\overline{\omega}_1+\ldots)}{\partial z}-\frac{\partial(\overline{\omega}_0+q\overline{\omega}_1+\ldots)}{\partial r}\cdot\frac{\partial\Im(\overline{\omega}_0+q\overline{\omega}_1+\ldots)}{\partial r}$$
$$=\left(\frac{\partial(\Im\overline{\omega}_0+\overline{\omega}_0)}{\partial(r,z)}\right)+q^1\left(\frac{\partial(\Im\overline{\omega}_1+\overline{\omega}_0)}{\partial(r,z)}+\frac{\partial(\Im\overline{\omega}_0+\overline{\omega}_1)}{\partial(r,z)}\right) \qquad (40)$$
$$+q^2\left(\frac{\partial(\Im\overline{\omega}_2+\overline{\omega}_0)}{\partial(r,z)}+\frac{\partial(\Im\overline{\omega}_0+\overline{\omega}_2)}{\partial(r,z)}\frac{\partial(\Im\overline{\omega}_1+\overline{\omega}_1)}{\partial(r,z)}\right)+\ldots..,$$

$$\frac{\partial(\overline{\omega}_0+\sigma^1\overline{\omega}_1+\ldots)}{\partial z}\cdot\Im(\overline{\omega}_0+\sigma^1\overline{\omega}_1+\ldots)=q^0\left(\Im\overline{\omega}_0\frac{\partial\overline{\omega}_0}{\partial z}\right)$$
$$+q^1\left(\Im\overline{\omega}_1\frac{\partial\overline{\omega}_0}{\partial z}+\Im\overline{\omega}_0\frac{\partial\overline{\omega}_1}{\partial z}\right)+q^2\left(\Im\overline{\omega}_2\frac{\partial\overline{\omega}_0}{\partial z}+\Im\overline{\omega}_1\frac{\partial\overline{\omega}_1}{\partial z}+\Im\overline{\omega}_0\frac{\partial\overline{\omega}_2}{\partial z}\right)+\ldots.., \qquad (41)$$

Using (40) and (41) in (38) we get

$$\Im\omega-\Im\Phi_0+\Im\overline{\omega}_0-q\Im^{-1}\left[\left(\frac{\partial(\Im\overline{\omega}_0+\overline{\omega}_0)}{\partial(r,z)}\right)+q^1\left(\frac{\partial(\Im\overline{\omega}_1+\overline{\omega}_0)}{\partial(r,z)}+\frac{\partial(\Im\overline{\omega}_0+\overline{\omega}_1)}{\partial(r,z)}\right)+\right.$$
$$q^2\left(\frac{\partial(\Im\overline{\omega}_2+\overline{\omega}_0)}{\partial(r,z)}+\frac{\partial(\Im\overline{\omega}_0+\overline{\omega}_2)}{\partial(r,z)}+\frac{\partial(\Im\overline{\omega}_1+\overline{\omega}_1)}{\partial(r,z)}\right)+\ldots$$
$$\left.-q^0\left(\Im\omega_0\frac{-2}{r^2}\frac{\partial\omega_0}{\partial z}\right)-q^1\left(\Im\omega_1\frac{-2}{r^2}\frac{\partial\omega_0}{\partial z}+\Im\omega_0\frac{-2}{r^2}\frac{\partial\omega_1}{\partial z}\right)+\ldots,\right]=0$$

We first define the inverse linear operator $\Im^{-2}$, consider equation (3), $\Im\Phi=0$.

Then

$$\Im\Phi=\Im_{rr}\Phi-\frac{1}{r}\Im_r\Phi+\Im_{zz}\Phi \quad\Rightarrow\quad \Im_{rr}\Phi-\frac{1}{r}\Im_r\Phi+\Im_{zz}\Phi=0, \qquad \Im_{rr}\Phi+\Im_{zz}\Phi=\frac{1}{r}\Im_r\Phi$$

Multiplying both sides of the above equation by $\mathbf{r}$, we get

$$\Im_r \Phi = r\Im_{rr}\Phi + r\Im_{zz}\Phi \tag{42}$$

In similar way, we get

$$\Im_{zz}\Phi = \frac{1}{r}\Im_r \Phi - \Im_{rr}\Phi \tag{43}$$

$$\Im_{rr}\Phi = \frac{1}{r}\Im_r \Phi - \Im_{zz}\Phi \tag{44}$$

$$\begin{pmatrix} \Phi = \psi_1 + \Im_r^{-1}\left(r\Im_{rr}\Phi + r\Im_{zz}\Phi\right) \\ \Phi = \psi_2 + \Im_{zz}^{-1}\left(\frac{1}{r}\Im_r \Phi - \Im_{rr}\Phi\right) \\ \Phi = \psi_3 + \Im_{rr}^{-1}\left(\frac{1}{r}\Im_r \Phi - \Im_{zz}\Phi\right) \end{pmatrix} \tag{45}$$

$$\Phi = \frac{\psi_1 + \psi_2 + \psi_3}{3} + \frac{1}{3}\left(\Im_r^{-1}\left(r\Im_{rr}\Phi + r\Im_{zz}\Phi\right) + \Im_{zz}^{-1}\left(\frac{1}{r}\Im_r \Phi - \Im_{rr}\Phi\right) + \Im_{rr}^{-1}\left(\frac{1}{r}\Im_r \Phi - \Im_{zz}\Phi\right)\right), \tag{46}$$

where $\Phi_0 = \frac{\psi_1 + \psi_2 + \psi_3}{3}$, ψ_1, ψ_2 and ψ_3 are the solutions of homogenous equations $\Im_r \Phi = 0$, $\Im_{rr}\Phi = 0$ and $\Im_{zz}\Phi = 0$ then

$$\Phi = \Phi_0 + \frac{1}{3}\left(\Im_r^{-1}\left(r\Im_{rr}\Phi + r\Im_{zz}\Phi\right) + \Im_{zz}^{-1}\left(\frac{1}{r}\Im_r \Phi - \Im_{rr}\Phi\right) + \Im_{rr}^{-1}\left(\frac{1}{r}\Im_r \Phi - \Im_{zz}\Phi\right)\right), \tag{47}$$

The inverse linear operators $\Im_r^{-1}$, $\Im_{zz}^{-1}$ and $\Im_{rr}^{-1}$ are defined by

$$\Im_r^{-1}\int(.)dr, \qquad \Im_{zz}^{-1}\int(.)dzdz, \qquad \Im_{rr}^{-1}\int(.)drdr. \tag{48}$$

Then we have

$$\begin{aligned}
\Phi_1 &= \frac{1}{3}\left(\Im_r^{-1}\left(r\Im_{rr}+r\Im_{zz}\right)+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)\Phi_0\\
\Phi_2 &= \frac{1}{3}\left(\Im_r^{-1}\left(r\Im_{rr}+r\Im_{zz}\right)+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)\Phi_1\\
&\vdots=\vdots\\
\Phi_{n+1} &= \frac{1}{3}\left(\Im_r^{-1}\left(r\Im_{rr}+r\Im_{zz}\right)+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)\Phi_n
\end{aligned} \tag{49}$$

Let us denote the quantity within the brackets by ℓ, then the following expression is obtained

$$\begin{bmatrix}
\Phi_1 = \frac{1}{3}\ell\Phi_0\\
\Phi_2 = \frac{1}{3}\ell\Phi_1 = \frac{1}{3}\ell\frac{1}{3}\ell\Phi_0 = \frac{1}{3^2}\ell^2\Phi_0,\\
\Phi_3 = \frac{1}{3}\ell\Phi_2 = \frac{1}{3^3}\ell^3\Phi_0\\
\vdots=\vdots\\
\Phi_{n+1} = \frac{1}{3^{n+1}}\ell^{n+1}\Phi_0
\end{bmatrix} \tag{50}$$

$$\Phi = \sum_{n=0}^{\infty}\frac{1}{3^n}\ell^n\Phi_0 = \sum_{n=0}^{\infty}\frac{1}{3^n}\left(\Im_r^{-1}(r\Im_{rr}+\mathrm{r}\Im_{zz})+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)^n\Phi_n$$

Thus the inverse linear operator can be easily identified as

$$\Im^{-2} = \left(\Im_{rr}-\frac{1}{r}\Im_r-\Im_{zz}\right)^{-2} = \sum_{n=0}^{\infty}\frac{1}{3^n}\left(\left(\Im_r^{-1}\left(r\Im_{rr}+r\Im_{zz}\right)\right)+\Im_{zz}^{-1}\left(\frac{1}{r}\Im_r-\Im_{rr}\right)+\Im_{rr}^{-1}\left(\frac{1}{r}\Im_r-\Im_{zz}\right)\right)^n \tag{51}$$

Now the zeroth order problem is $\Im\overline{\omega_0} = \Im\Phi_0 \qquad \Rightarrow \overline{\omega_0} = \Phi_0$ (52)

The 1st order problem is

$$\Im\varpi_1 + \Im\varpi_0 = v^{-1}\Im^{-1}\left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\varpi_0,\varpi_0)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}.\Im\varpi_0\right)$$

Operating with $\Im$ both sides of the above equations

$$\Im^2\varpi_1 + \Im^2\varpi_0 = v^{-1}\left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\varpi_0,\varpi_0)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}.\Im\varpi_0\right)$$

In order to find the initial guess of HPM, we make use of the zeroth order problem as $\Im^2\varpi_0 = 0$

$$\Im^2\varpi_1 = v^{-1}\left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\varpi_0,\varpi_0)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}.\Im\varpi_0\right)$$

operating $\Im^{-2}$ on both sides to get $\varpi_1 = v^{-1}\Im^{-2}\left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\varpi_0,\varpi_0)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}.\Im\varpi_0\right)$ (53)

The 2nd order problem is:

$$\varpi_2 = v^{-1}\Im^{-2}\left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\varpi_1,\varpi_0)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\varpi_0}{\partial z}.\Im\varpi_1 + \frac{\partial(\Im\varpi_0,\varpi_1)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\varpi_1}{\partial z}.\Im\varpi_0\right)$$

and so on. The series solution form of the problem reads as, $\Phi = \lim_{\vartheta\to 1}\varpi(r,z;q) = \varpi_1 + \varpi_2 + \varpi_3 + \cdots$

Where the following quantities are defined $\Phi_0 = \frac{\Psi_1+\Psi_2+\Psi_3}{3}$

$$\Phi_1 = v^{-1}\Im^{-2}\left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\Phi_0,\Phi_0)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}.\Im\Phi_0\right)$$

$$\Phi_2 = v^{-1}\Im^{-2}\left(\frac{\partial(\Im\Phi_1,\Phi_0)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}.\Im\Phi_1 + \frac{\partial(\Im\Phi_0,\Phi_1)}{\partial(\mathrm{r},z)} - \frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}.\Im\Phi_0\right) \qquad (54)$$

and so on. The series form of the solution can be written as

$$\Phi = \sum_{n=0}^{\infty}\Phi_n$$

$$\Phi = \Phi_0 + v^{-1}\Im_1^{-2}\left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\Phi_0,\Phi_0)}{\partial(\mathrm{r},z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) - 2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_0}{\partial z^2} - \Im_1^{-2}\frac{\partial^4\Phi_0}{\partial z^4} +$$

$$v^{-1}\Im_1^{-2}\left[\left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\Phi_1,\Phi_0)}{\partial(\mathrm{r},z)} - \Im\Phi_1\frac{2}{r^2}\frac{\partial\Phi_0}{\partial z}\right) + \left(\frac{1}{\mathrm{r}}\frac{\partial(\Im\Phi_0,\Phi_1)}{\partial(\mathrm{r},z)} - \Im\Phi_0\frac{2}{r^2}\frac{\partial\Phi_1}{\partial z}\right)\right] + \cdots -$$

$$2\Im_1^{-2}\frac{\partial^2\Im_1\Phi_1}{\partial z^2} - \Im_1^{-2}\frac{\partial^4\Phi_1}{\partial z^4} + \cdots$$

3. Conclusion

In this paper we consider two methods which are the Adomian decomposition and the Homotopy perturbation method. A two states for the available linear operators and obtained the approximation for our problem have been considered. We can see that for the first state, the available linear operator was split in two parts and for the second case, we considered the full linear form of the operator without splitting it into parts.

The present analysis can be applied to a wide range of the physical and engineering problems. As compared to the Adomian decomposition method for the analysis of the problem, we have the great advantage of the selection of the initial guess which can be chosen on the basis of the previous knowledge, and most importantly, the initial approximation should satisfy the given initial or boundary conditions, which leads us to the uniformly valid approximately series solution.

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